

COMS21202: Symbols, Patterns and Signals

Problem Sheet 3: Probabilistic Models

1. You were consulted by a Physics student who is trying to estimate the voltage (V) given current (I) and resistance (R) information. The student informs you that the physical model is

$$I = \frac{V}{R} + \epsilon$$

subject to a measurement error $\epsilon \sim \mathcal{N}(0, \sigma^2)$. Assuming *i.i.d* observations, the likelihood of $p(D|V)$ where V is the model's only parameter and D is the observed data is equal to:

$$p(D|V) = \prod_i \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{(I_i - \frac{V}{R_i})^2}{\sigma^2}}$$

where I_i is the current value for observation i and R_i is the resistance value for observation i . Prove, using the Maximum Likelihood Estimation (MLE) recipe, that

$$V_{ML} = \sum \frac{I_i}{R_i} / \sum \frac{1}{R_i^2}$$

Answer:

$$1. \text{ Take the natural logarithm: } \ln p(D|V) = N \ln \frac{1}{\sqrt{2\pi}\sigma} + \sum_i -\frac{1}{2} \frac{(I_i - \frac{V}{R_i})^2}{\sigma^2}$$

$$2. \text{ Take the derivative: } \frac{d}{dV} \ln p(D|V) = \frac{1}{\sigma^2} \sum_i \frac{1}{R_i} (I_i - \frac{V}{R_i})$$

$$3. \text{ Find the solution by setting derivative to 0: } -\frac{1}{\sigma^2} \sum_i \frac{1}{R_i} (I_i - \frac{V}{R_i}) = 0$$

$$\sum_i \frac{I_i}{R_i} - V_{ML} \sum_i \frac{1}{R_i^2} = 0$$

$$V_{ML} = \sum \frac{I_i}{R_i} / \sum \frac{1}{R_i^2}$$

2. For a given probabilistic model,

$$p(D|\theta) = b e^{-(3-\theta)^2}$$

where b is a normalising constant and a known prior of

$$p(\theta) = c e^{-\theta(\theta-1)}$$

where c is a normalising constant, Find the maximum a posteriori value of the model parameters θ . **Clearly show the steps you followed in finding the answer.**

Answer:

$$p(D|\theta)p(\theta) = b e^{-(3-\theta)^2} c e^{-\theta(\theta-1)} \quad (1)$$

$$\ln p(D|\theta)p(\theta) = \ln b + \ln c - (3-\theta)^2 - \theta(\theta-1) \quad (2)$$

$$\frac{d}{d\theta} \ln p(D|\theta)p(\theta) = 2(3-\theta) - 2\theta + 1 \quad (3)$$

$$= -4\theta + 7 \quad (4)$$

$$-4\theta_{MAP} + 7 = 0 \quad (5)$$

$$\theta_{MAP} = \frac{7}{4} \quad (6)$$

3. Suppose that X is a discrete random variable with the following probability mass function, where $0 \leq \theta \leq 1$ is a parameter.

X	0	1	2	3
$P(X)$	$\frac{2\theta}{3}$	$\frac{\theta}{3}$	$\frac{2(1-\theta)}{3}$	$\frac{(1-\theta)}{3}$

The following 10 independent observations were taken from this distribution:

3	0	2	1	3	2	1	0	2	1
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- (a) What is the Maximum Likelihood estimate of θ
 (b) Assume you have prior knowledge that $p(\theta) = b\theta(1-\theta)$, what would the Maximum a Posteriori (MAP) be?

Answer:

- (a) *Following the MLE recipe,*

$$\begin{aligned} p(D|\theta) &= \prod_i P(d_i|\theta) \\ &= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2 \\ &= c\theta^5(1-\theta)^5 \text{ (where } c \text{ is a constant)} \end{aligned}$$

1. *Take the natural log, so*

$$\ln p(D|\theta) = \ln c + 5 \ln \theta + 5 \ln(1-\theta)$$

2. *Take the derivative*

$$\frac{d}{d\theta} \ln p(D|\theta) = \frac{5}{\theta} - \frac{5}{1-\theta}$$

3. *Set the derivative to 0*

$$\begin{aligned} \frac{5}{\theta_{ML}} - \frac{5}{1-\theta_{ML}} &= 0 \\ 5(1-\theta_{ML}) - 5\theta_{ML} &= 0 \\ \theta_{ML} &= \frac{1}{2} \end{aligned}$$

- (b) *When introducing prior,*

$$\begin{aligned} p(D|\theta)p(\theta) &= \left(\frac{2\theta}{3}\right)^2 \left(\frac{\theta}{3}\right)^3 \left(\frac{2(1-\theta)}{3}\right)^3 \left(\frac{1-\theta}{3}\right)^2 b\theta(1-\theta) \\ &= c\theta^6(1-\theta)^6 \text{ (where } c \text{ is a constant)} \end{aligned}$$

Following the same formula as in (a), $\theta_{MAP} = 0.5$ (same as θ_{ML} for this particular case)