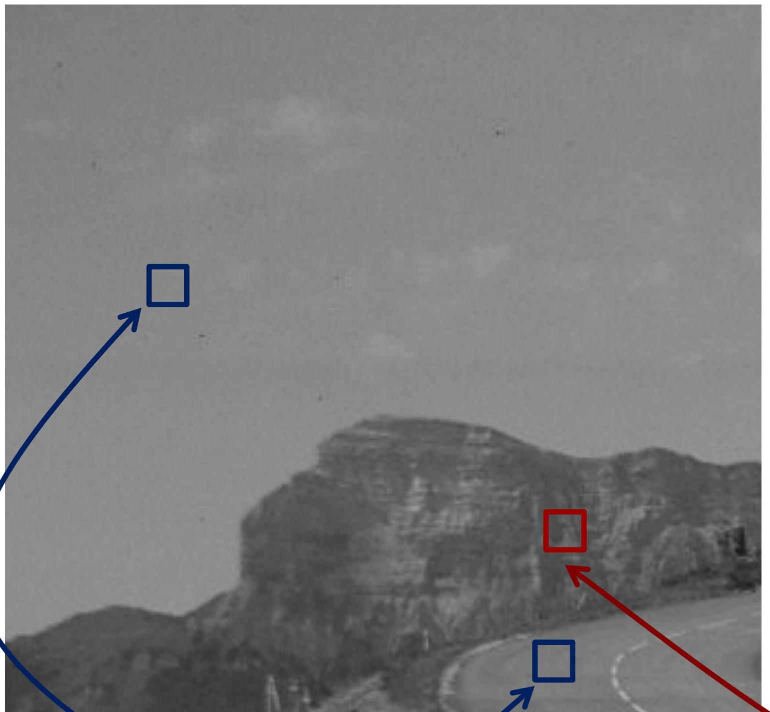
COMS21202: Symbols, Patterns and Signals

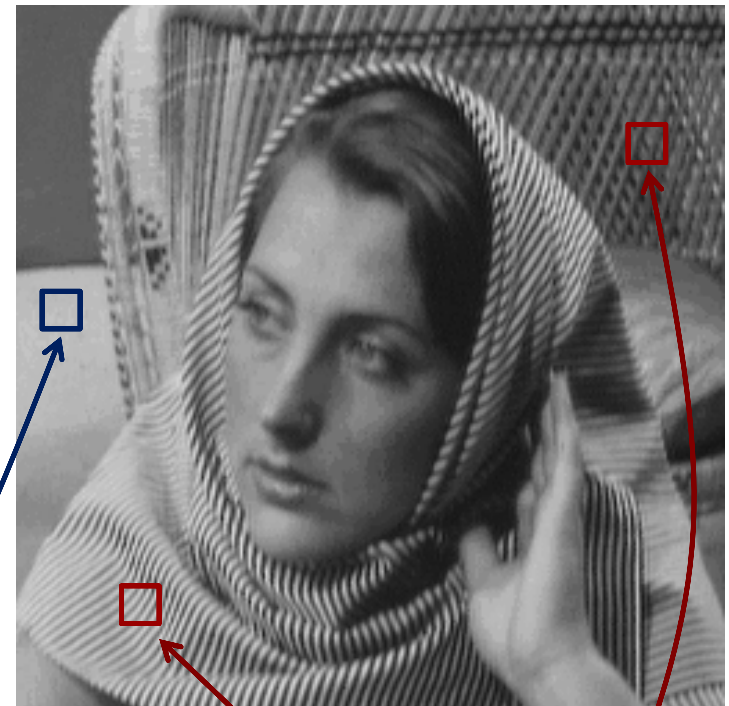
2D FT and Spatial Frequency

Fourier Transform → straightforward extension to 2D.

- Images are functions of two variables → e.g. $f(x,y)$
- Defined in terms of *spatial frequency* → 2D frequency.
- Fourier Transform is particularly useful for characterising this intensity variation across an image.
- *Rate of change of intensity* along each dimension.

Examples: Spatial Frequency



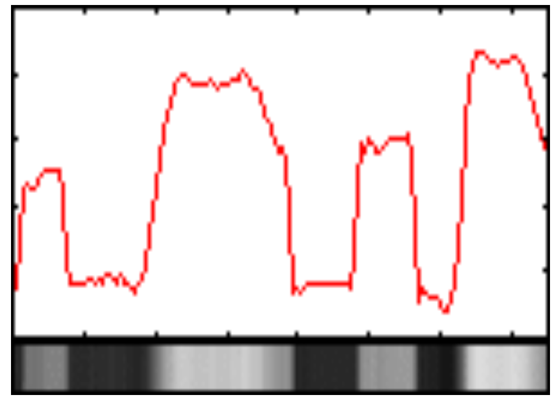


Slowly changing → low frequency

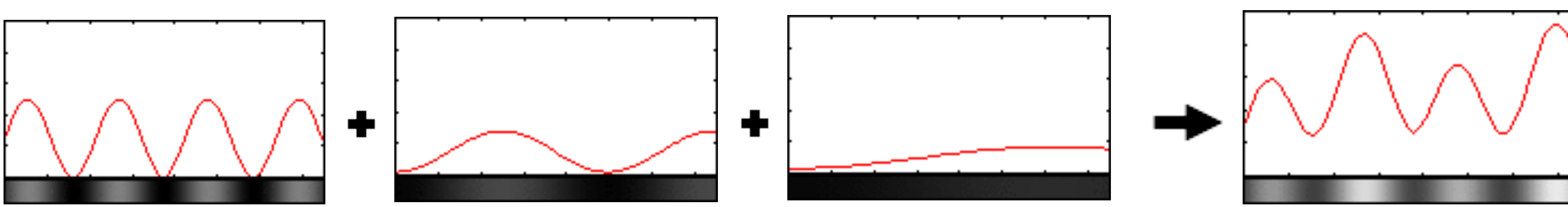
Rapidly changing →high frequency

Images are waves!? (or intuition behind FT)

Take a single row or column of pixel from an image, and graph it



Add some regular waves to get one that is close to (or as good as) the image



2D Fourier Transform: Continuous Form

- The Fourier Transform of a continuous function of two variables $f(x,y)$ is:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \; e^{-j2\pi(ux+vy)} \, dxdy$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse* Fourier Transform:

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) \; e^{j2\pi(ux+vy)} \, dudv$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform: Discrete Form

- The FT of a discrete function of two variables, $f(x,y)$, $x,y=0,1,2\ldots,N-1$, is:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y) \quad e^{-j2\pi\left(\frac{ux+vy}{N}\right)} \quad \text{for } u,v = 0,1,2,\ldots,N-1\,.$$

- Conversely, given $F(u,v)$, we can obtain $f(x,y)$ by means of the *inverse FT:*

$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} F(u,v) \quad e^{j2\pi\left(\frac{ux+vy}{N}\right)} \quad \text{for } x,y = 0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.

Note, they constitute a lossless representation of data.

2D Fourier Transform

- The concept of the frequency domain follows from Euler's Formula:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x,y)$ multiplied by sines and cosines of various frequencies:

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\left[\cos\left(\frac{2\pi(ux+vy)}{N}\right) - j\sin\left(\frac{2\pi(ux+vy)}{N}\right)\right]$$

$$\text{for } u,v = 0,1,2,\ldots,N-1\,.$$

We have transformed from a time domain to a frequency domain representation.

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when $u=0, v=0$ **1** **–** **0**

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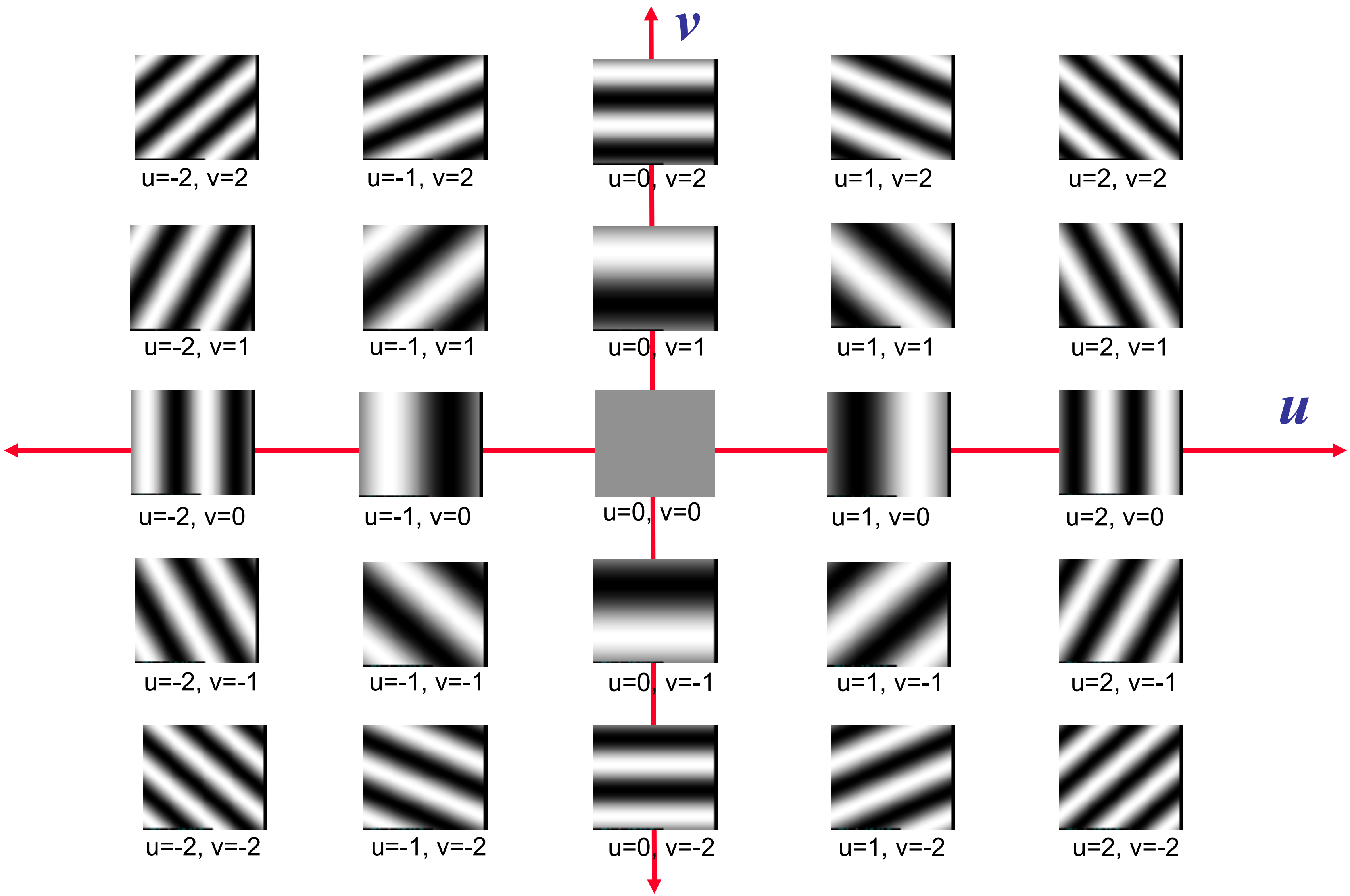
The slowest varying frequency component, i.e. when *u=0,v=0* → average image graylevel

We have transformed from a time domain to a frequency domain representation.

Another view: The 2D Basis Functions



2D Fourier Transform

- *F(u,v)* is a complex number & has real and imaginary parts:

$$F(u,v) = R(u,v) + jI(u,v)$$

- *Magnitude* or *spectrum* of the FT:

$$|F(u,v)| = \sqrt{R^2(u,v) + I^2(u,v)}$$

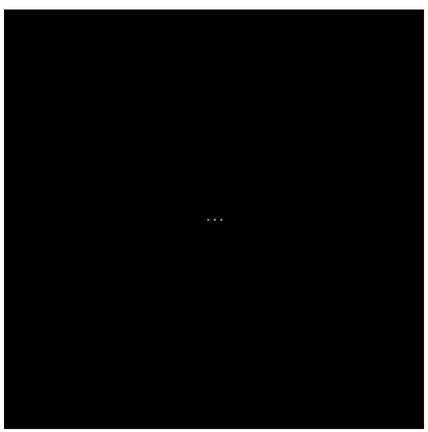
- Phase angle or phase spectrum:

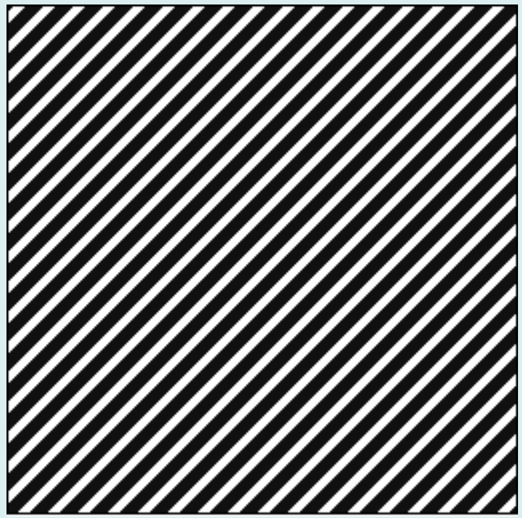
$$\varphi(u,v) = \tan^{-1}\frac{I(u,v)}{R(u,v)}$$

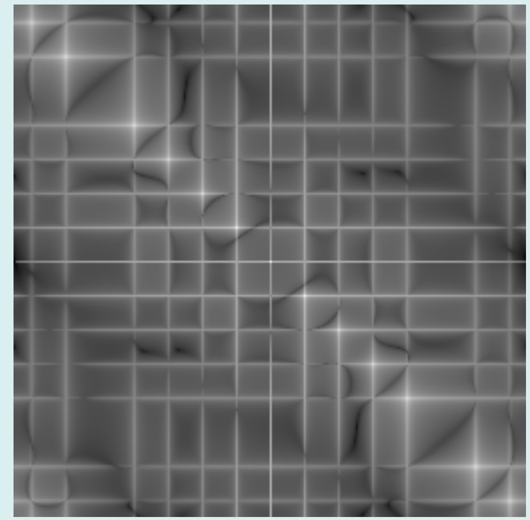
- Expressing *F(u,v)* in polar coordinates:

$$F(u,v) = |F(u,v)|e^{j\varphi(u,v)}$$

Example I: Image Analysis

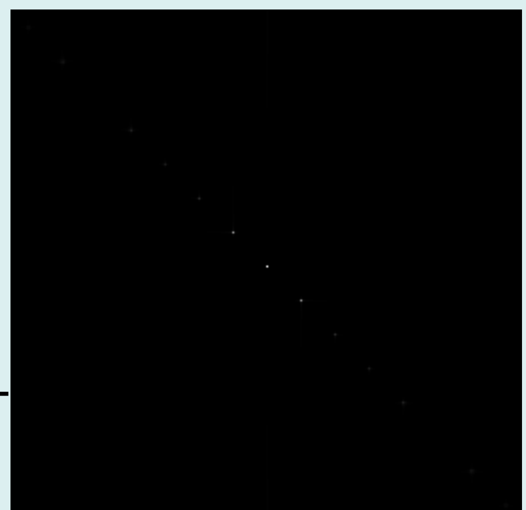


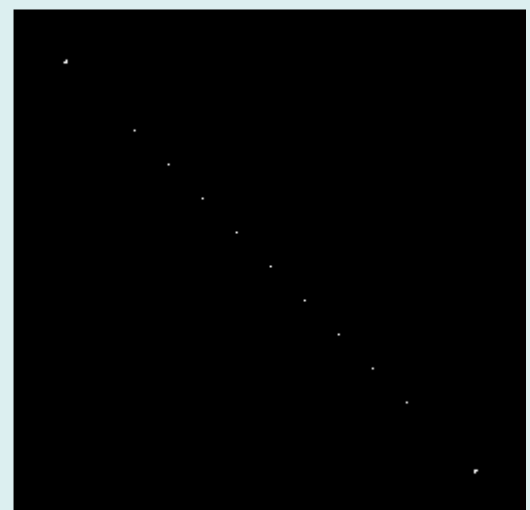




log of FT

FT

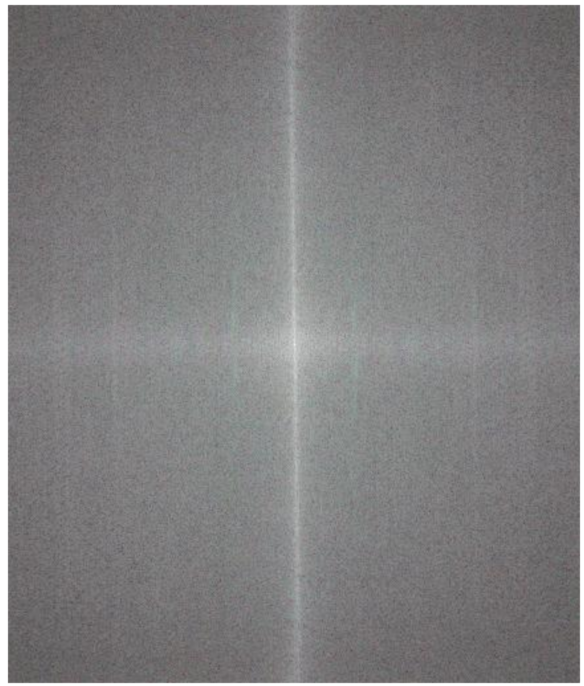




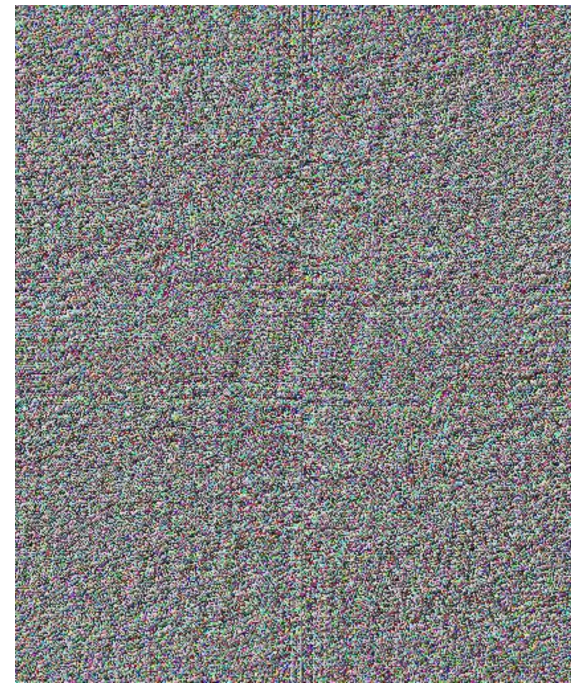
Thresholded log of FT

Example II: Magnitude + Phase

I



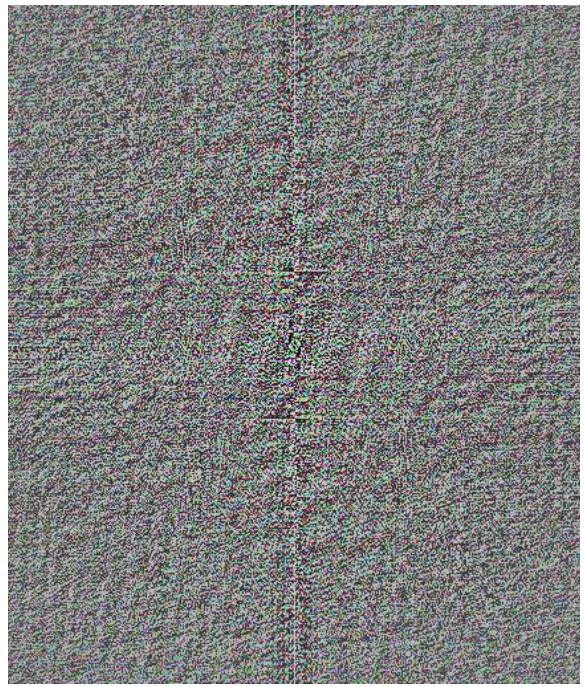
$\log(|F(I)|+1)$



$\angle[F(I)]$

Example III: Real + Imaginary

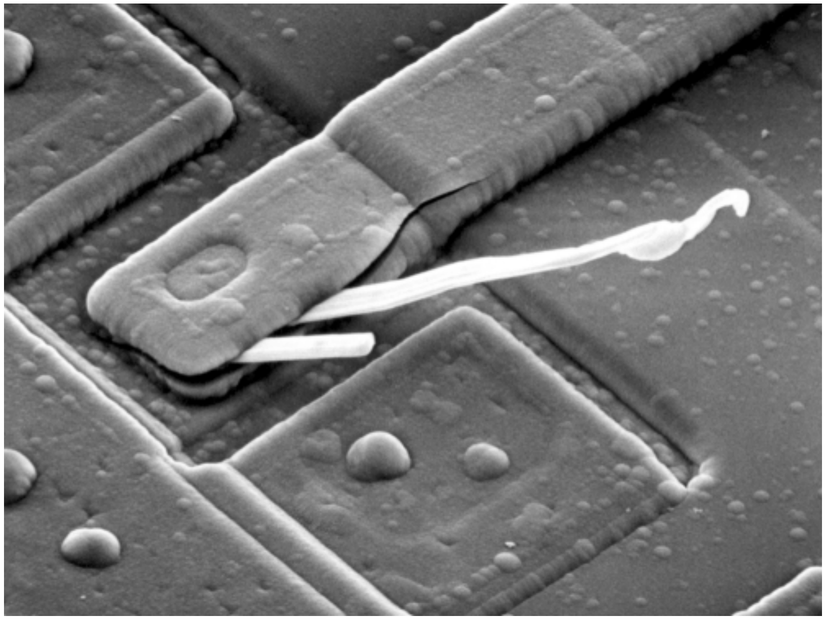
I



$\mathrm{Re}[\mathrm{F}(I)]$

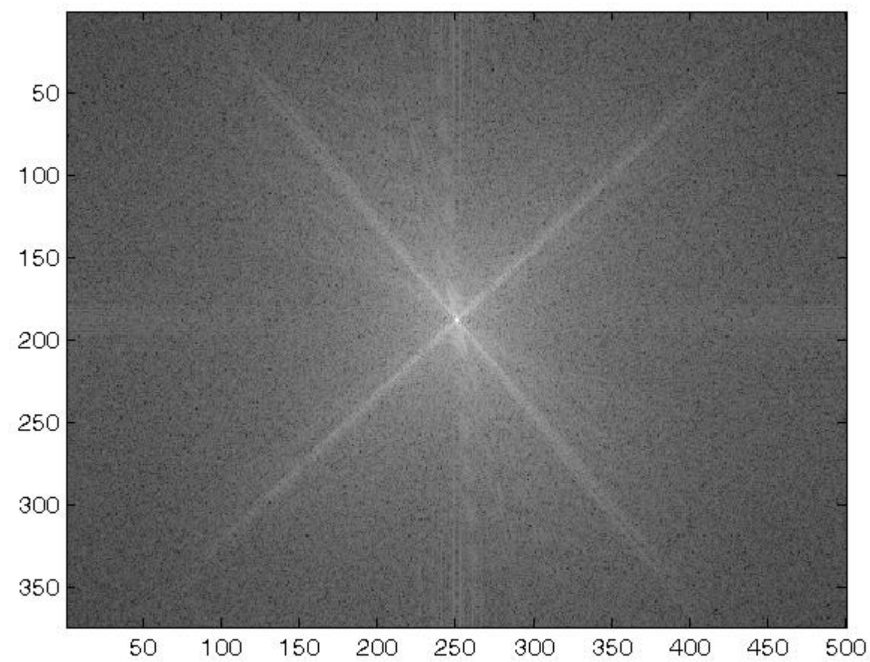
$\mathrm{Im}[\mathrm{F}(I)]$

Example IV: Interpreting the FS

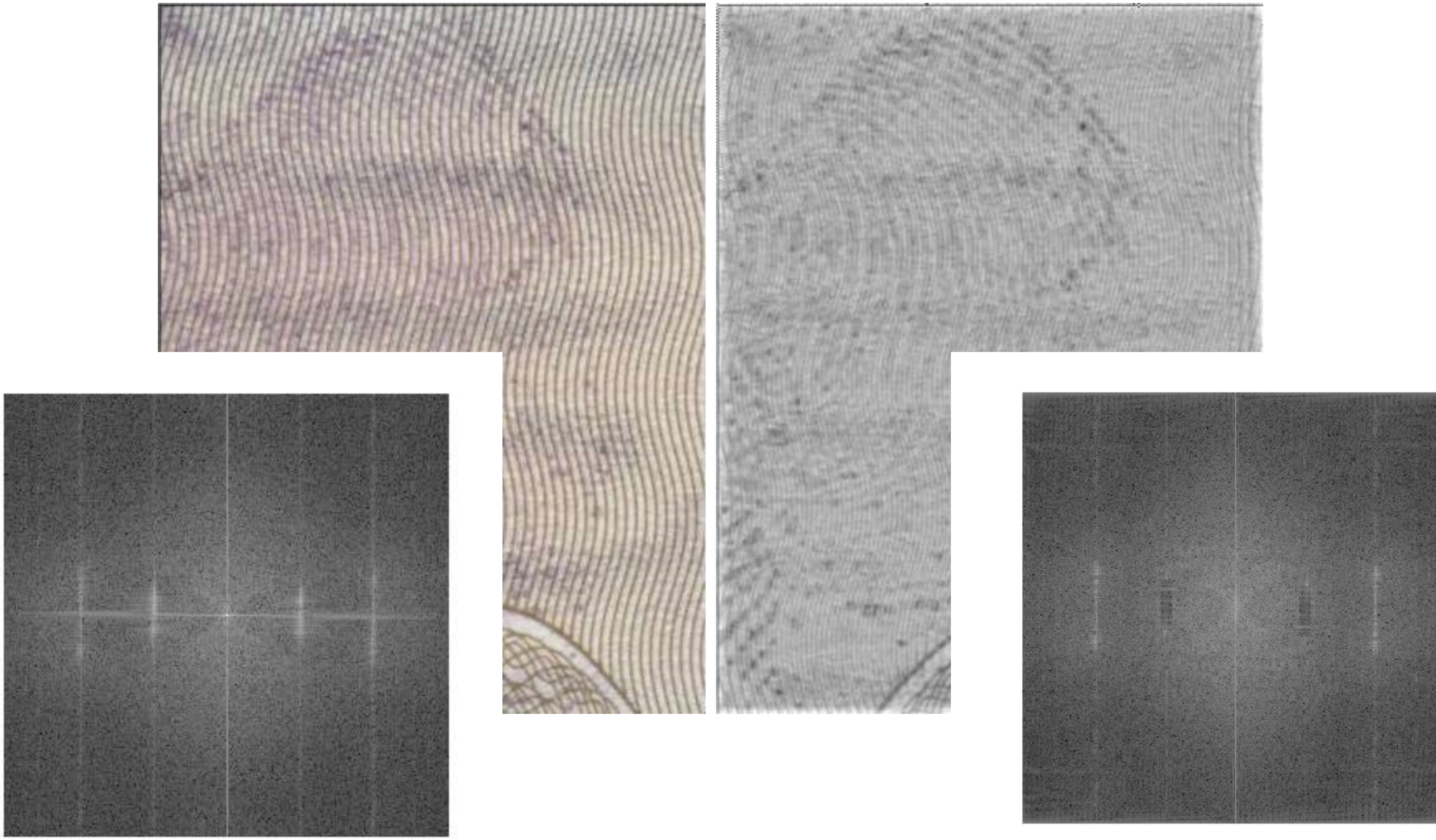


Scanning electron microscope image of an integrated circuit

Can we interpret what the bright components mean?



Example V: Image Analysis

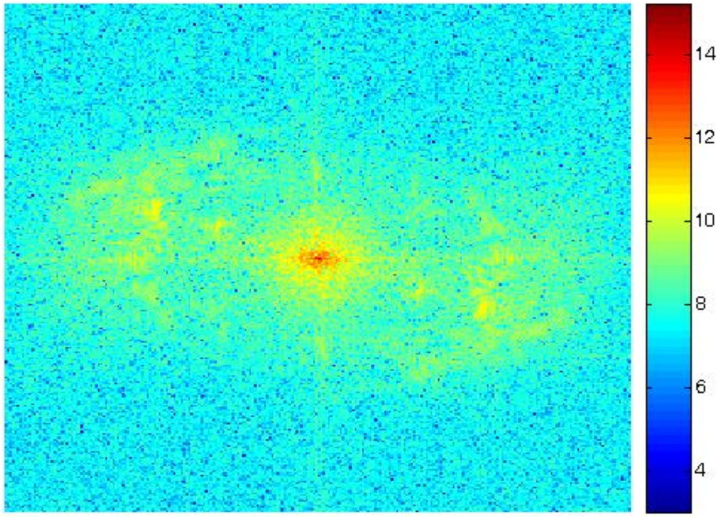


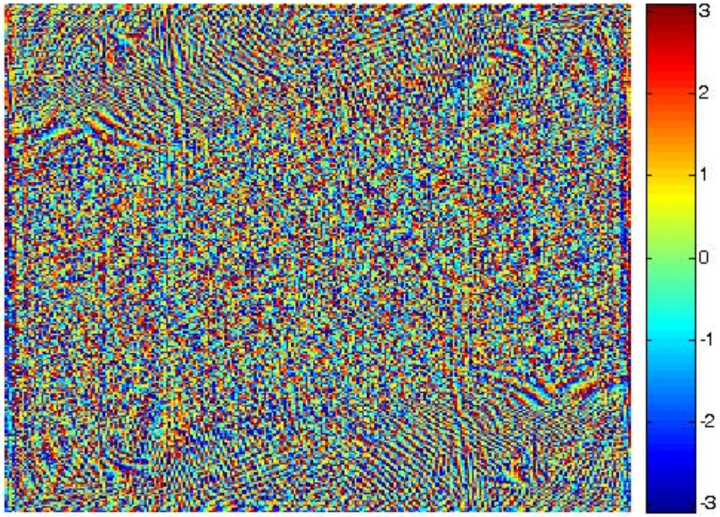
Matlab: 2D Fourier Transform

```
f = imread('barbara.gif');   %read in image
z = fft2(double(f));         % do fourier transform
q = fftshift(z);             % puts u=0,v=0 in the centre
Magq = abs(q);               % magnitude spectrum
Phaseq=angle(q);             % phase spectrum
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Usually for viewing purposes:
imagesc(log(abs(q)+1));
colorbar;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
w = ifft2(ifftshift(q));     % do inverse fourier transform
imagesc(w);
```

Matlab functions in RED

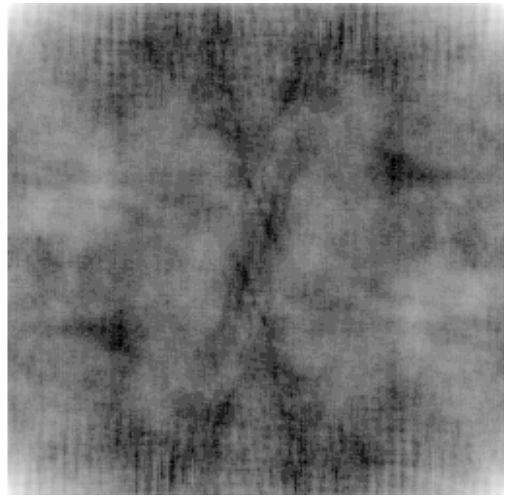
Viewing Magnitude and Phase



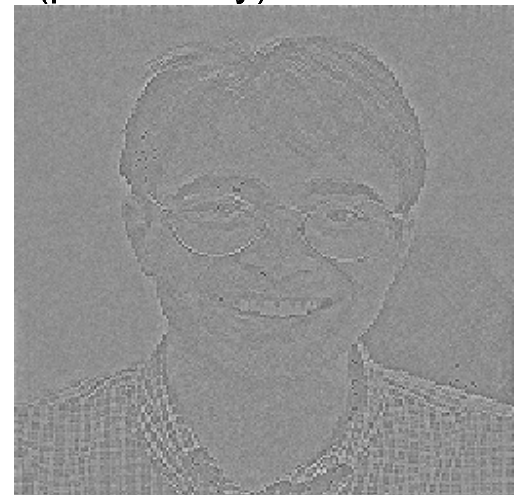


Importance of Phase

ifft(mag only)



ifft(phase only)



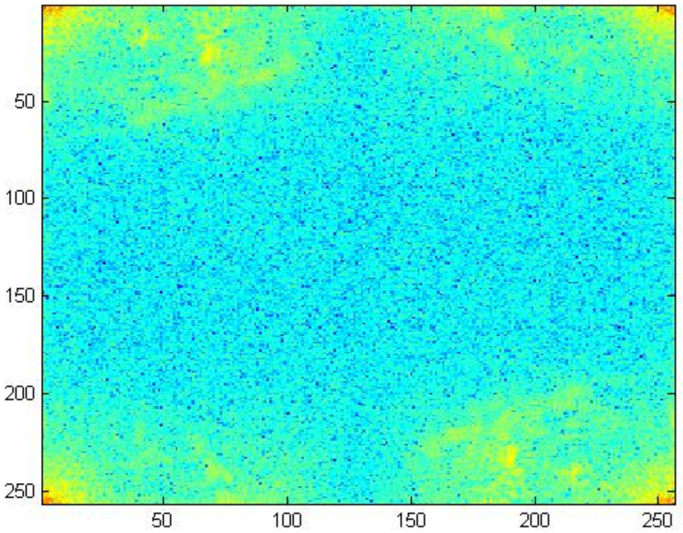
ifft(mag(Peter) and Phase(Andrew))

ifft(mag(Andrew) and Phase(Peter))

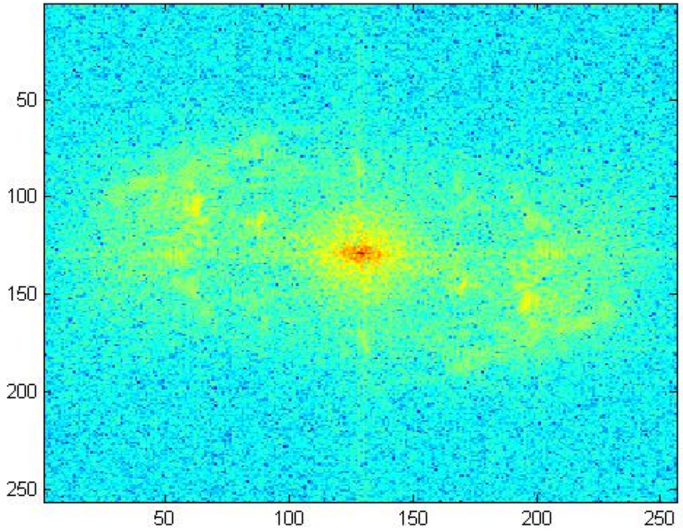
Periodic Spectrum

- Important property of the FT: *Conjugate Symmetry*
- The FT of a real function f(x,y) gives:

$$F(u,v) = F^*(-u,-v) \quad \Rightarrow \quad |F(u,v)| = |F^*(-u,-v)|$$



Before fftshift



After fftshift

Symmetry

- Important property of the FT: *Conjugate Symmetry*

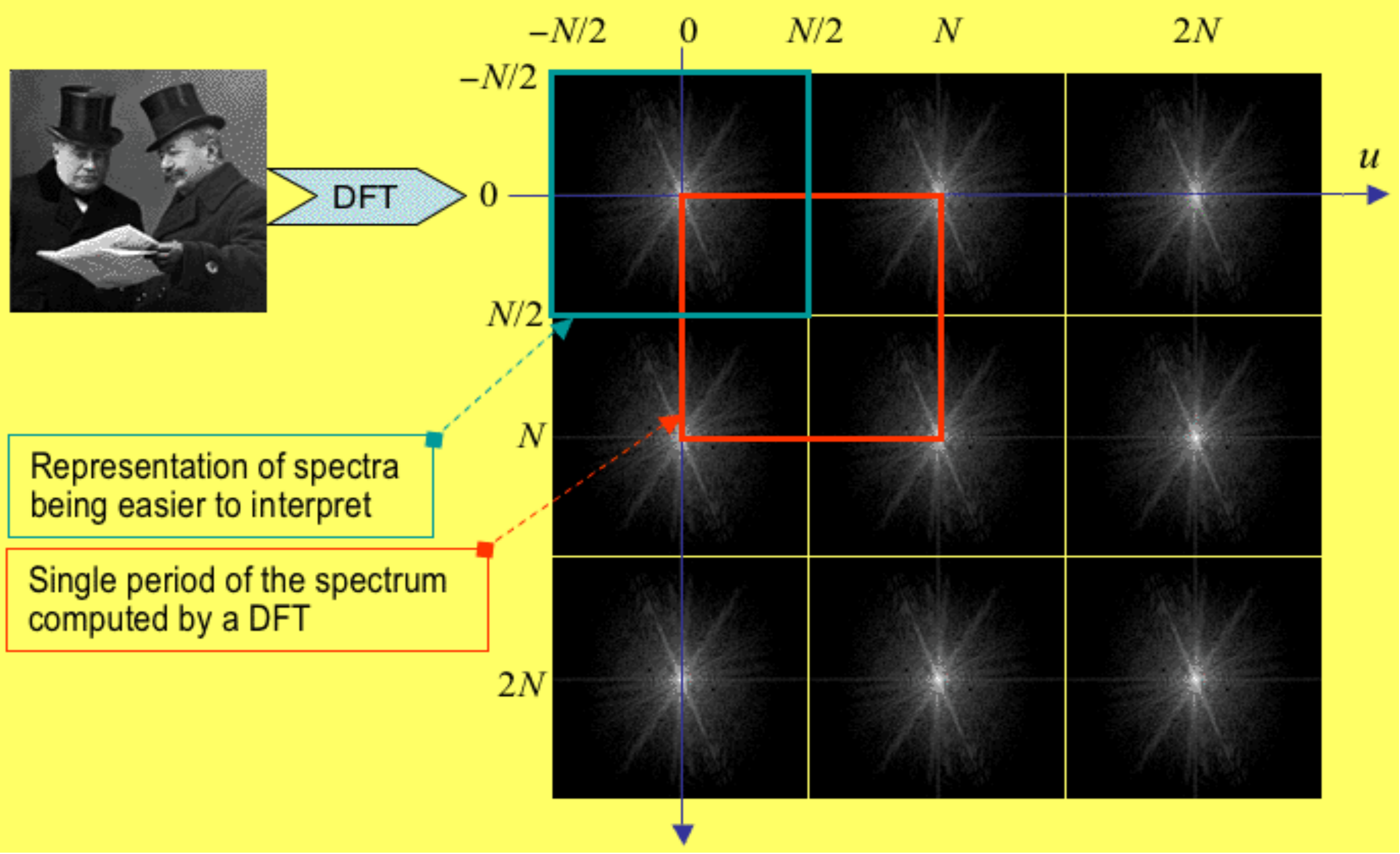


Illustration from Václav Hlaváč

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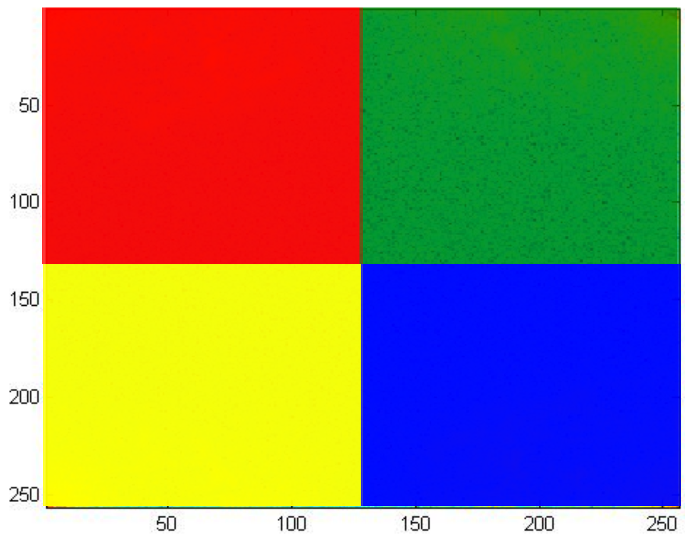
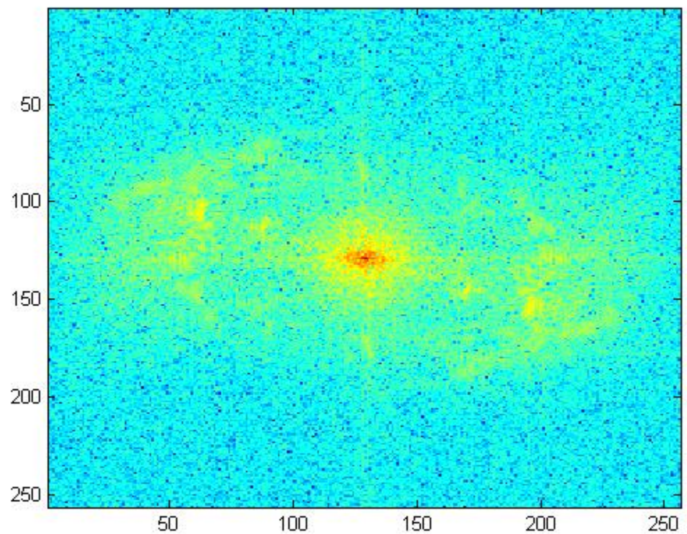


Before fftshift

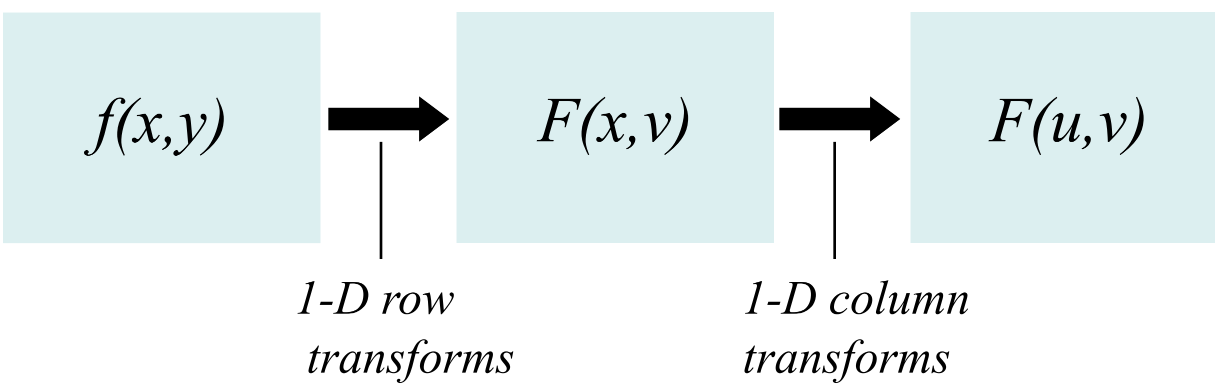


After fftshift

Separability

- Important property of the FT: *Separability*
- If a 2D transform is separable, the result can be found by successive application of two 1D transforms.

$$F(u,v) = \frac{1}{N}\sum_{x=0}^{N-1} F(x,v) \quad e^{\frac{-j2\pi ux}{N}} \text{ where } F(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y) \quad e^{\frac{-j2\pi vy}{N}}$$

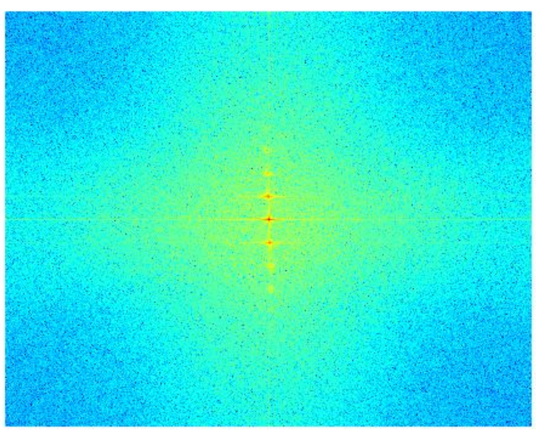


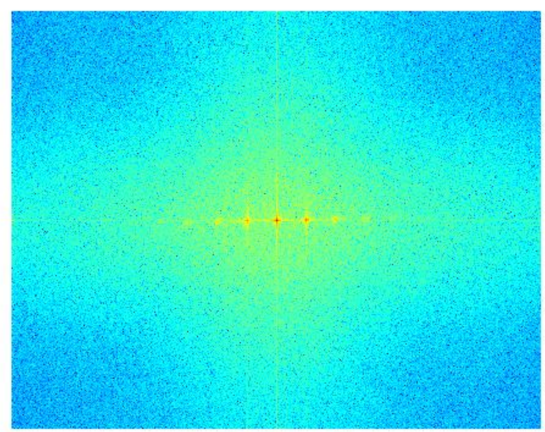
Rotation

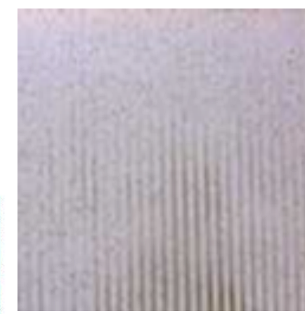
- Important property of the FT: *Rotation*
- Rotate the image and the Fourier space rotates.

$$x = r\cos\theta \quad y = r\sin\theta \qquad u = \omega\cos\varphi \quad v = \omega\sin\varphi$$

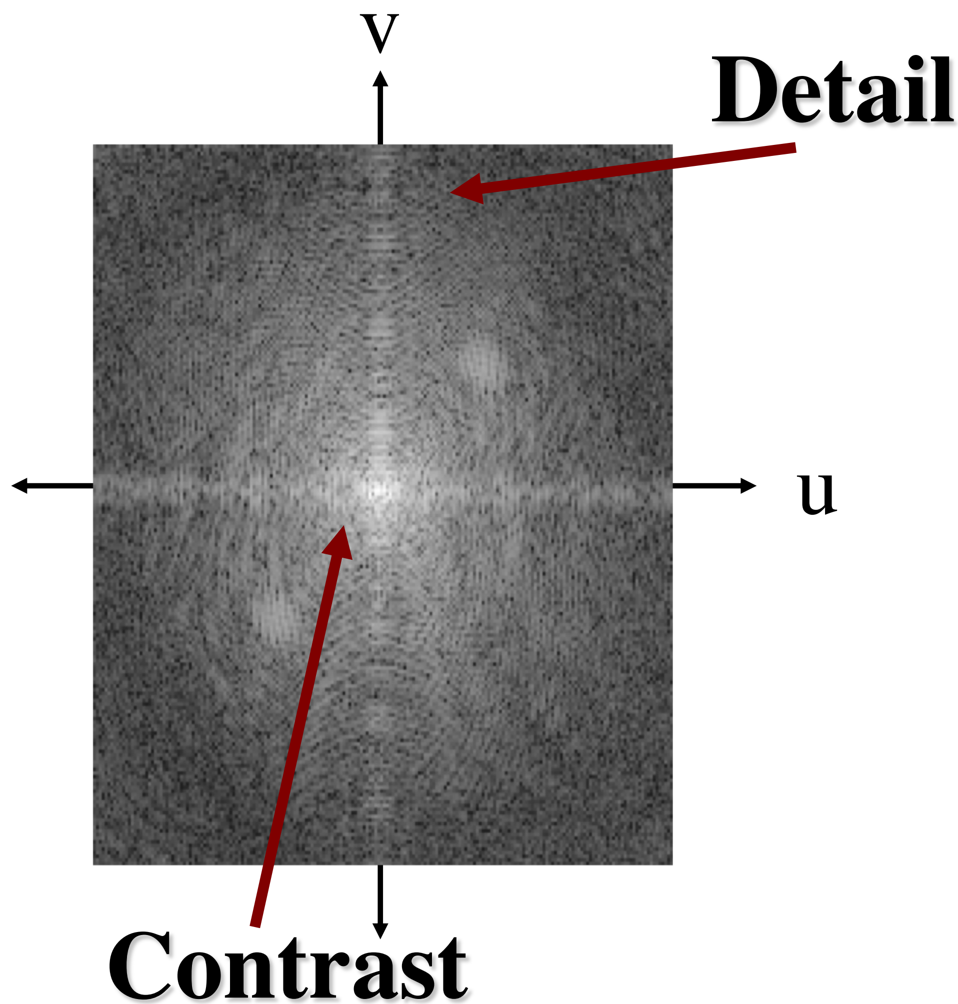
$$f(r, \theta + \theta_0) \qquad F(\omega, \varphi + \theta_0)$$

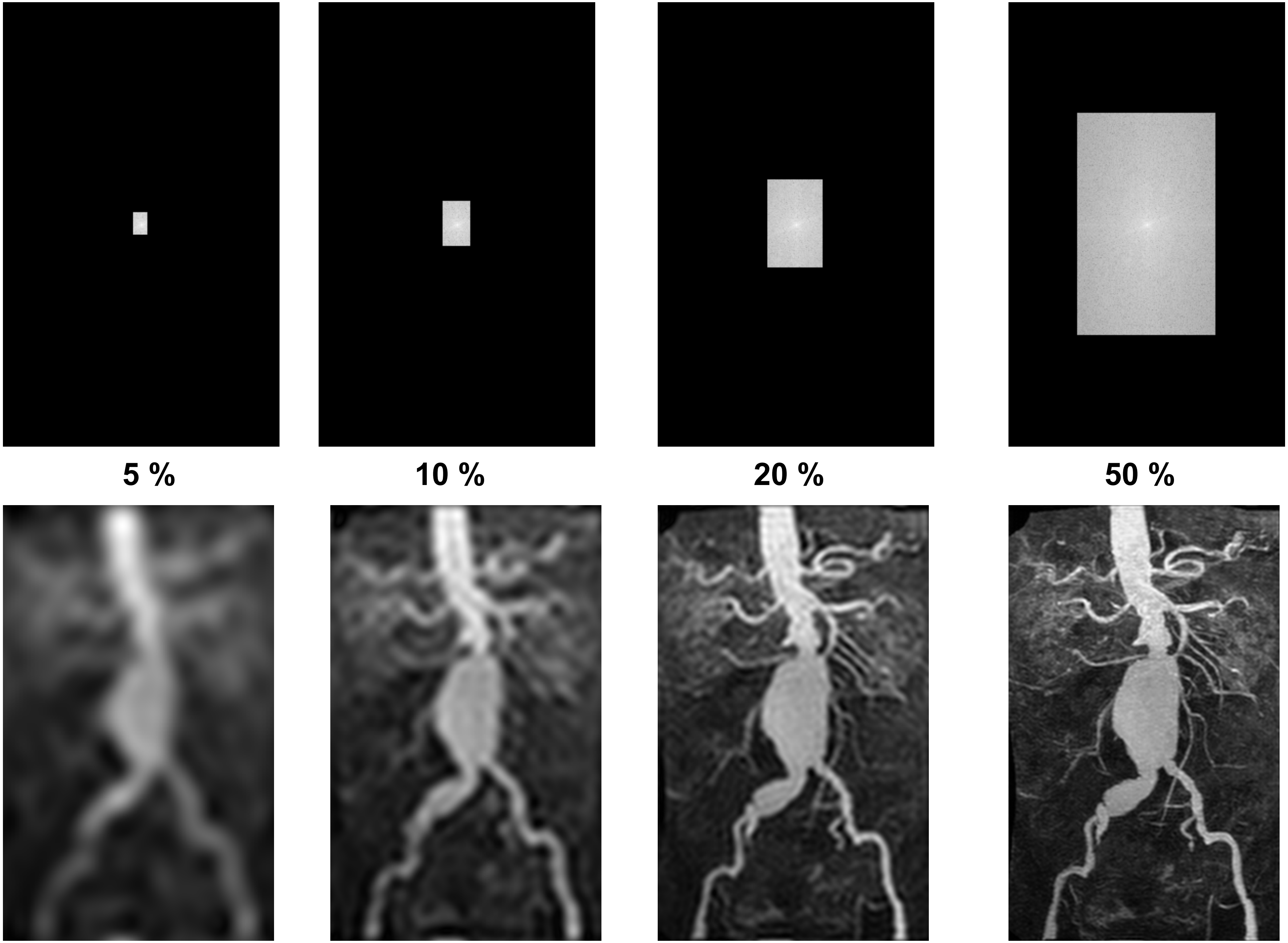






Manipulating the Fourier Frequencies



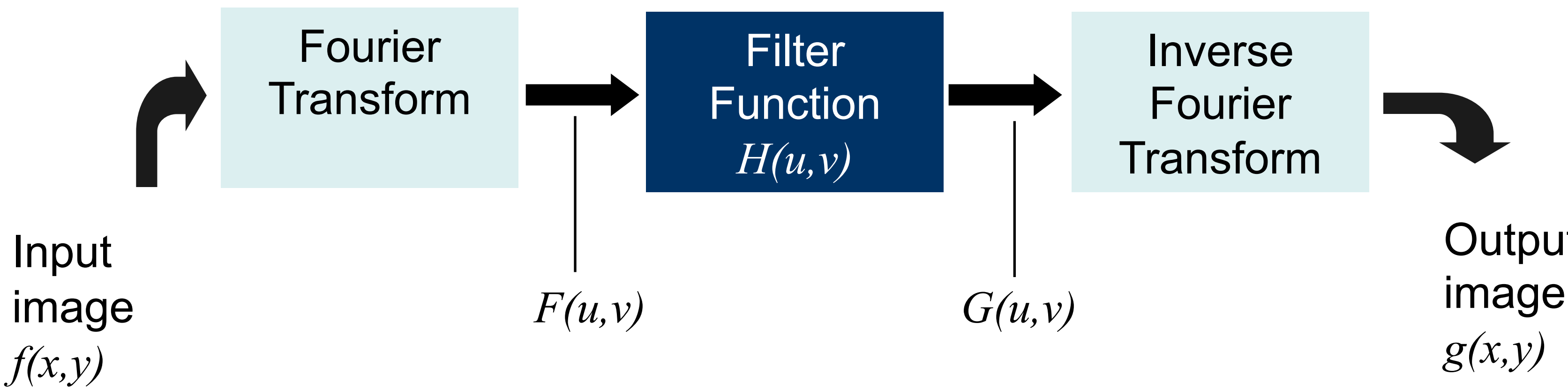

5 %
10 %
20 %
50 %

Filtering the Fourier Frequencies

- Filtering → to manipulate the (signal/image/etc) data.

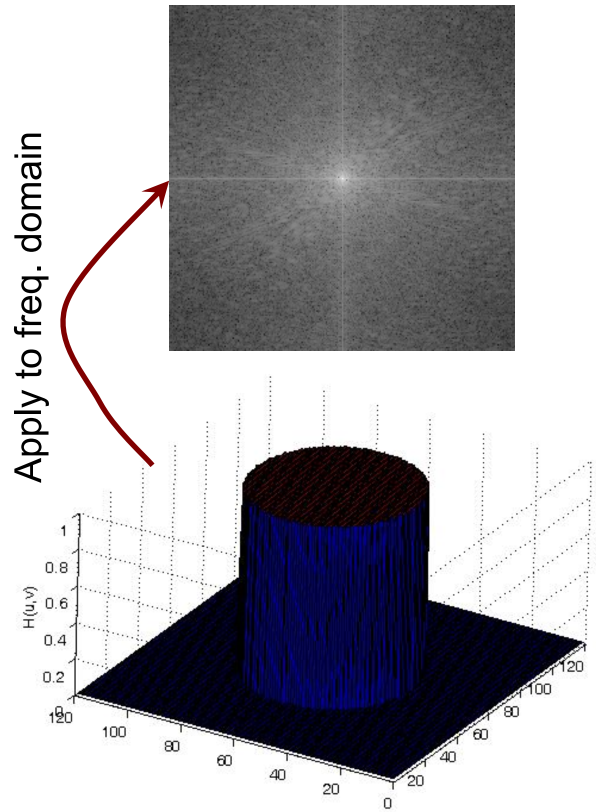
1D: $G(u) = F(u)H(u)$

2D: $G(u,v) = F(u,v)H(u,v)$



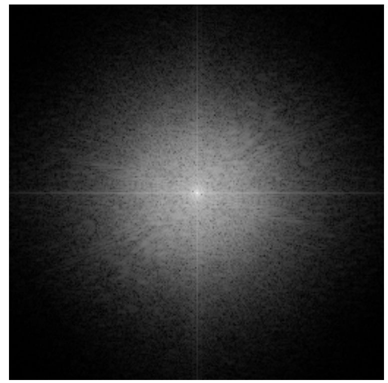
Low Pass Filtering

- 1D: turning the “treble” down on audio equipment!
- 2D: smooth image

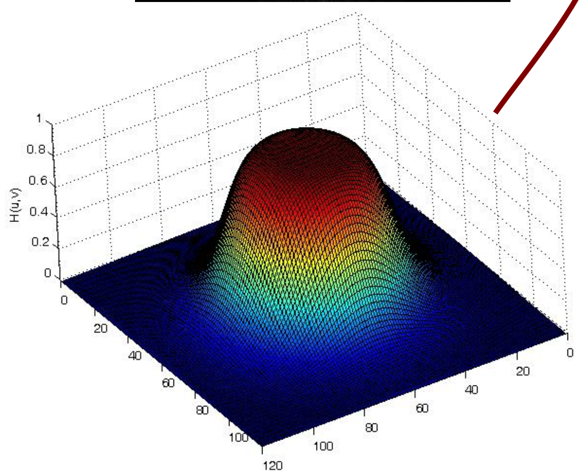


$$H(u,v) = \begin{cases} 1 & r(u,v) \le r_0 \\ 0 & r(u,v) > r_0 \end{cases} \qquad r(u,v) = \sqrt{u^2+v^2}, \; r_0 \text{ is the filter radius}$$

Butterworth’s Low Pass Filter



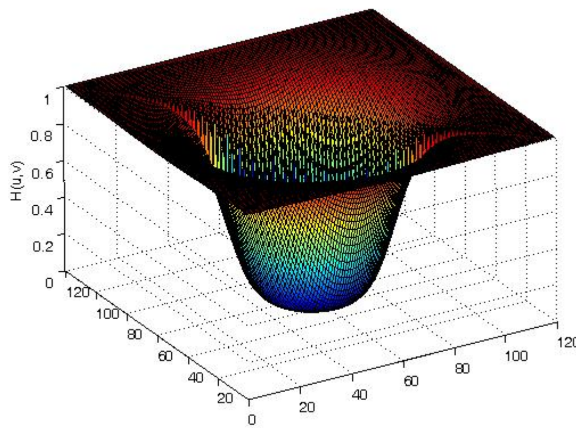
After applying to freq. domain



$$H(u,v) = \frac{1}{1 + [r(u,v)/r_0]^{2n}} \quad \text{of order } n$$

Butterworth's High Pass Filter

- 1D: turning the bass down on audio equipment!
- 2D: sharpen image



Order of $n=3$

$$H(u,v) = \frac{1}{1 + [r_0/r(u,v)]^{2n}} \quad \text{of order } n$$

Filtering to Remove Periodic Noise

- This is a very common application of the FT.