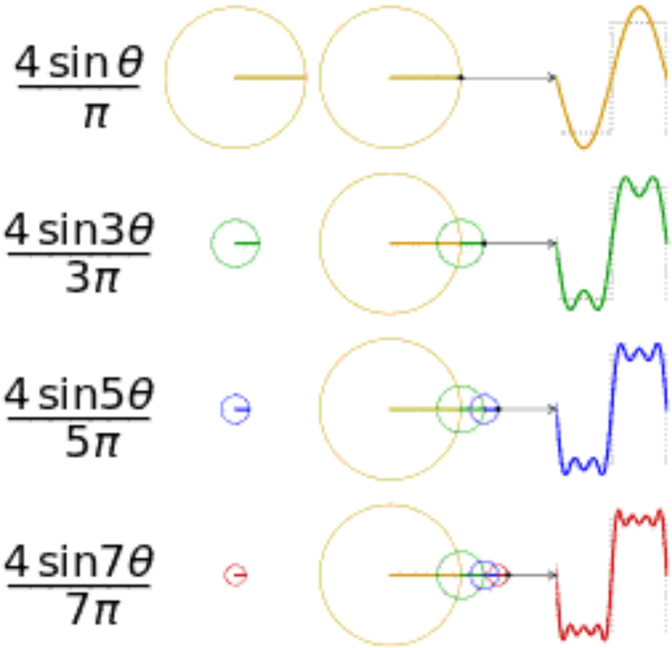
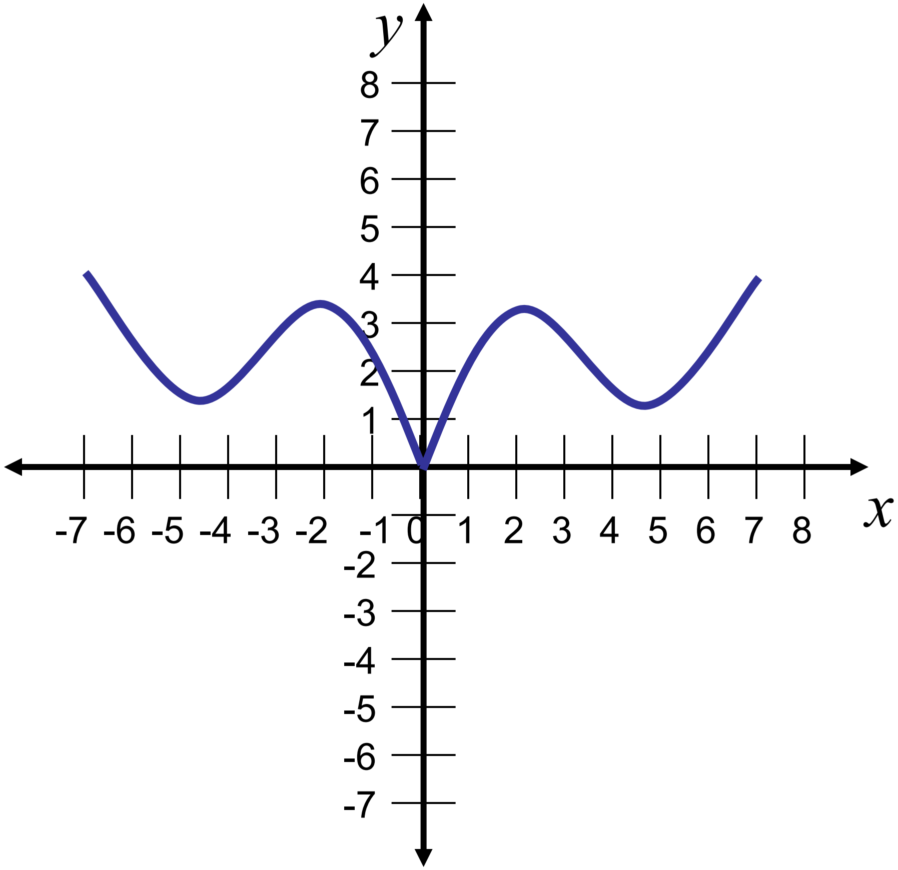
COMS21202: Symbols, Patterns and Signals

Signals and Fourier Analysis



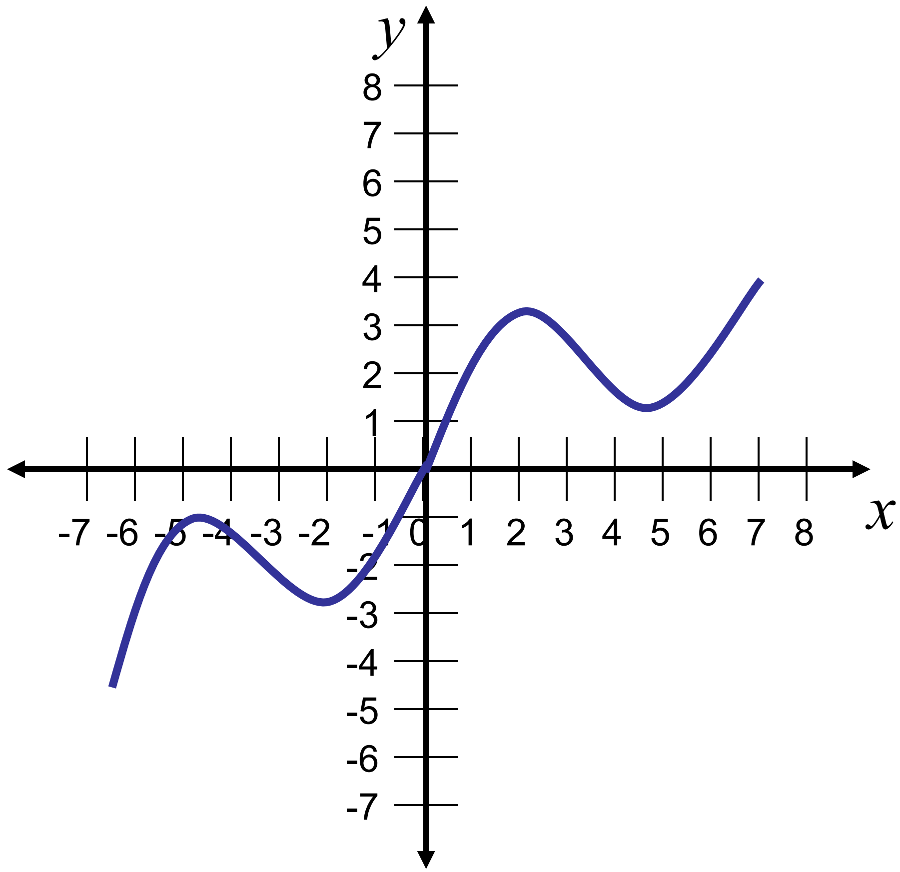
Animation from https://giphy.com

Reminder: Even Functions have y-axis Symmetry



So a function is **even** when $y = f(x) = f(-x)$

Reminder: Odd Functions have origin Symmetry



So a function is **odd** when $y = f(x) = -f(-x)$

Examples: odd or even?

$$f(x) = 3x^4 - 7x^2 + 1$$

Even

$$f(-x) = 3(-x)^4 - 7(-x)^2 + 1 = 3x^4 - 7x^2 + 1$$

$$f(x) = 4x^3 - x$$

Odd

$$f(-x) = 4(-x)^3 - (-x) = -4x^3 + x$$

$$f(x) = -3x^5 - x$$

Odd

$$f(-x) = -3(-x)^5 - (-x) = 3x^5 + x$$

$$f(x) = 5x^4 + 3x^2 + 1$$

Even

$$f(-x) = 5(-x)^4 + 3(-x)^2 + 1 = 5x^4 + 3x^2 + 1$$

Function Decomposition

Any function $f(x) \neq 0$ can be expressed as the sum of an even function $f_e(x)$ and an odd function $f_o(x)$.

$$f(x) = f_e(x) + f_o(x)$$

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

Even Part

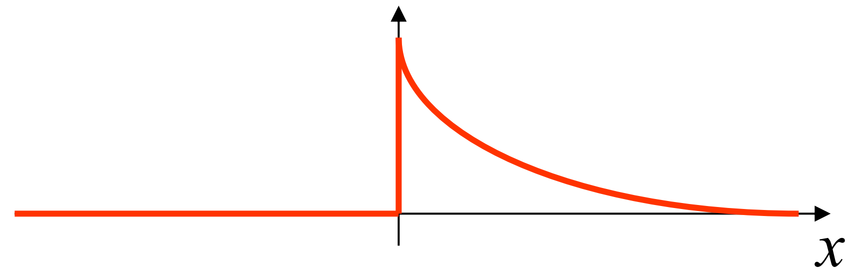
$$f_o(x) = \frac{1}{2}[f(x) - f(-x)]$$

Odd Part

Example

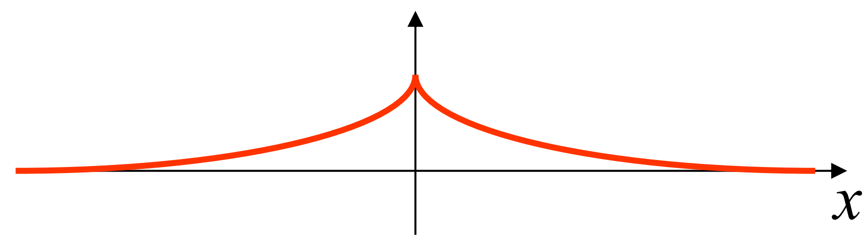
$$f(x) = f_e(x) + f_o(x)$$

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$



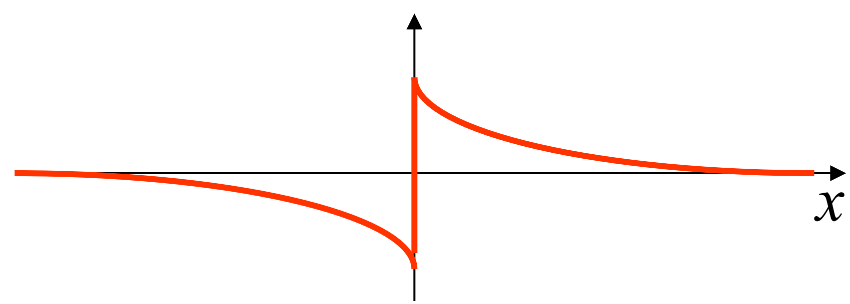
Even Part

$$f_e(x) = \begin{cases} \frac{1}{2} e^{-x} & x > 0 \\ \frac{1}{2} e^{x} & x < 0 \end{cases}$$

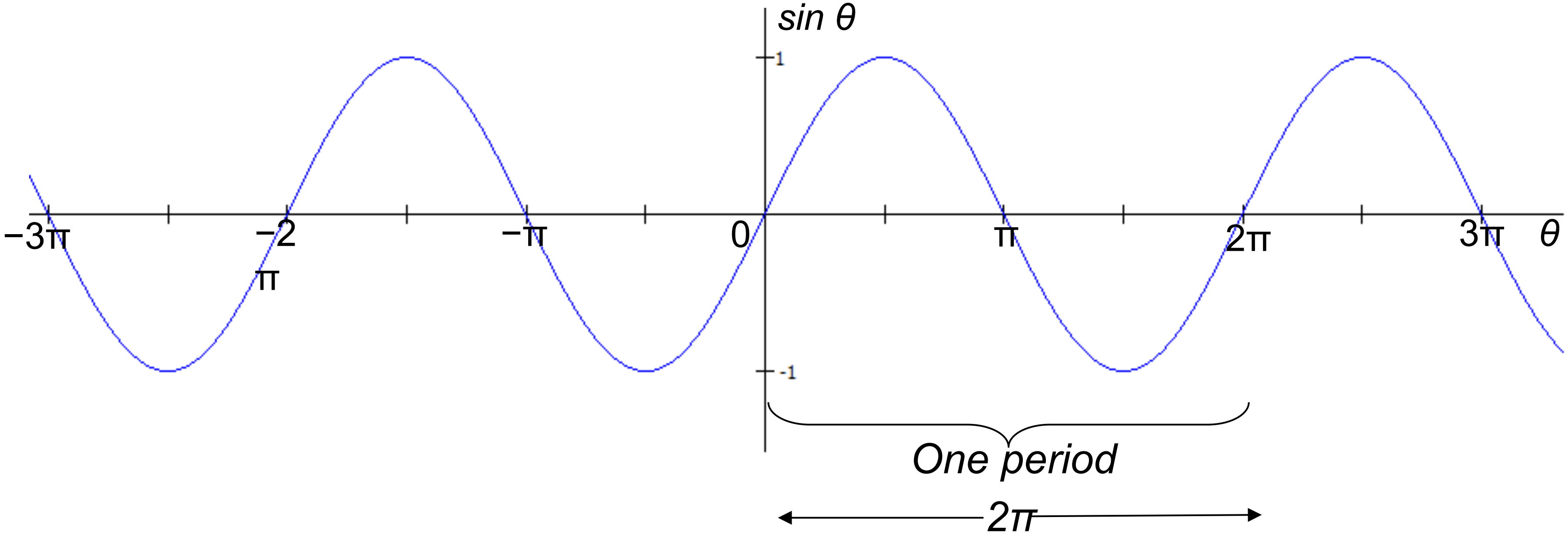


Odd Part

$$f_o(x) = \begin{cases} \frac{1}{2} e^{-x} & x > 0 \\ -\frac{1}{2} e^{x} & x < 0 \end{cases}$$

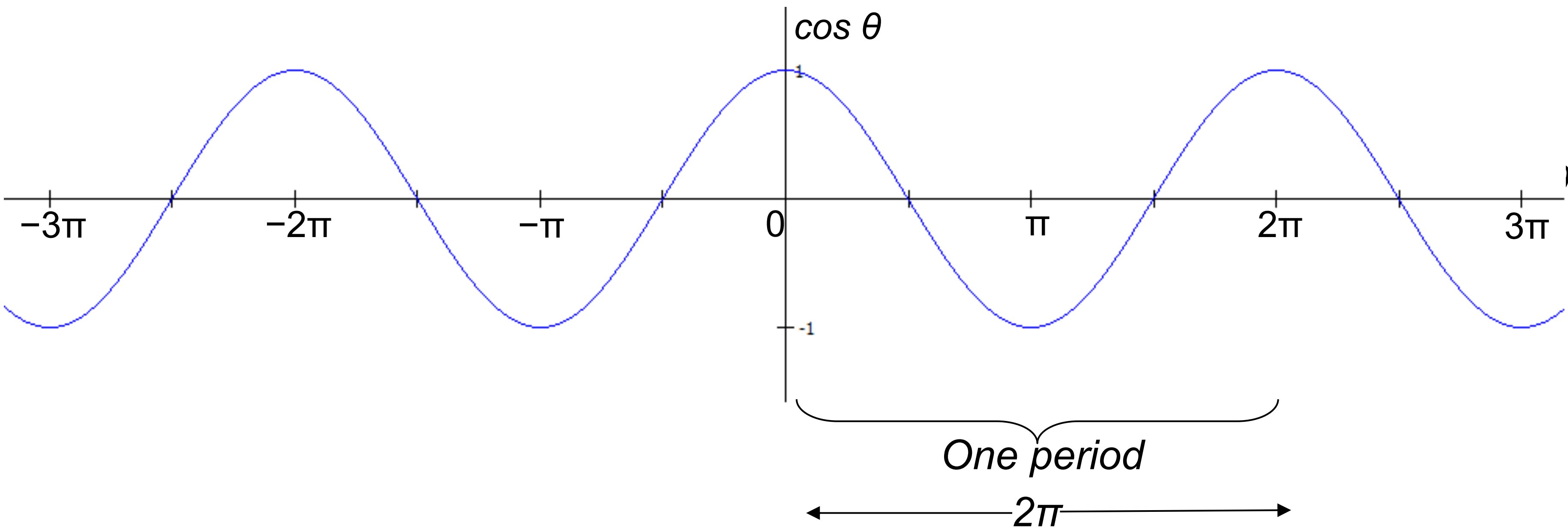


sine function



$sin\ \theta$ is an **odd** function as it is symmetric wrt the origin. $$sin(\theta) = -sin(-\theta)$$

cos function



$cos\ \theta$ is an **even** function as it is symmetric wrt to the y-axis. $cos(\theta) = cos(-\theta)$

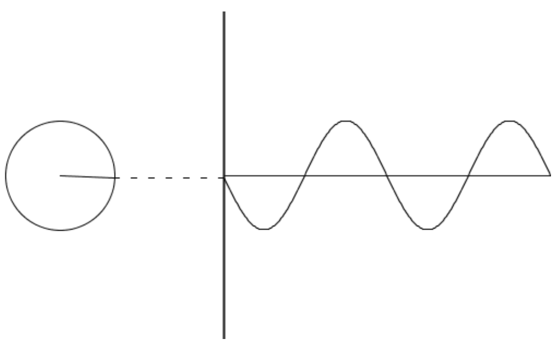
Signals as Functions

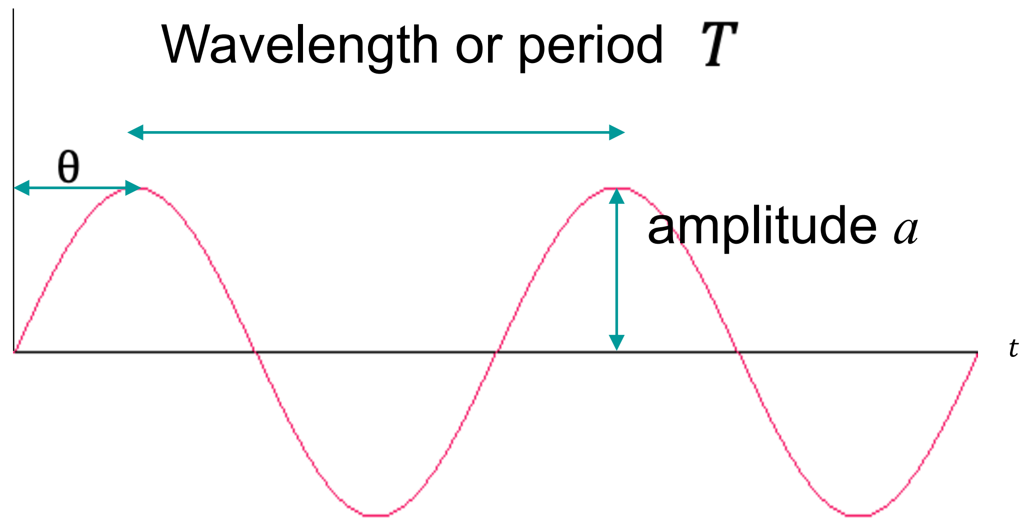
Frequency - allows us to characterise signals:

- Repeats over regular intervals with Frequency $u = \frac{1}{T}$ cycles/sec (Hz)
  - Amplitude a (peak value)
  - the Phase θ (shift in degrees)

Example: sine function

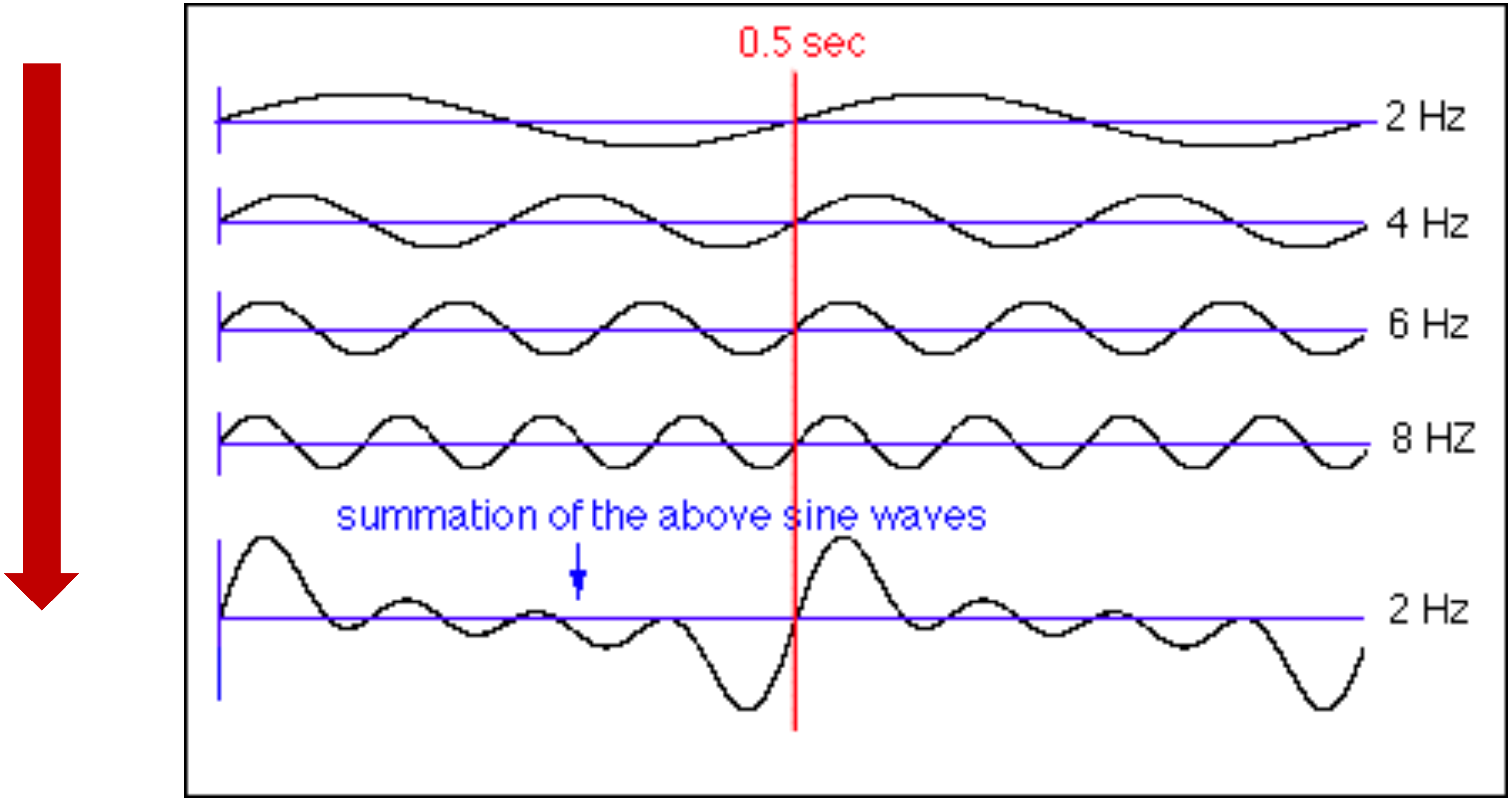
$$f(t) = a\, sin 2\pi u t$$





Reminder: Linear Systems

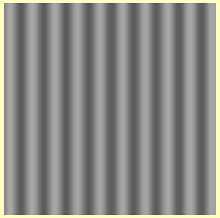
- *For a linear system, output of the linear combination of many input signals is the same linear combination of the outputs →superposition*

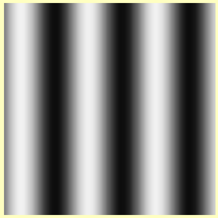


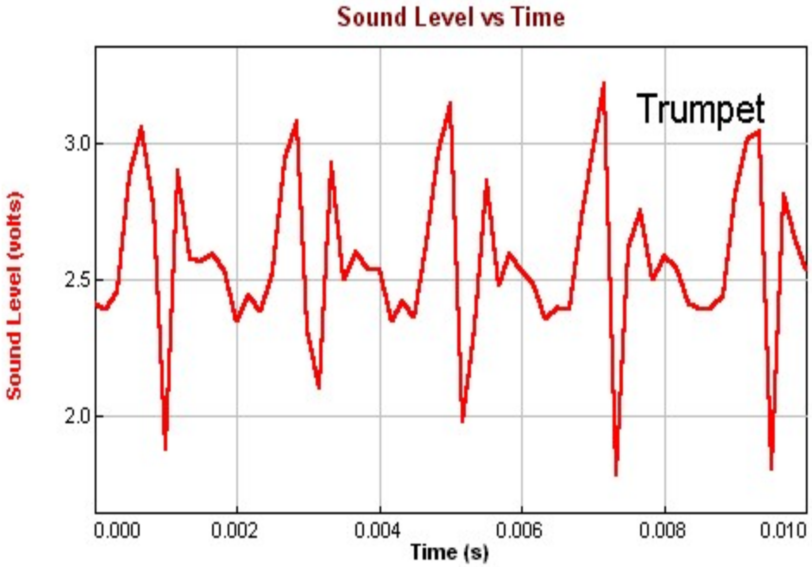
How do you interpret these musical instrument signals?

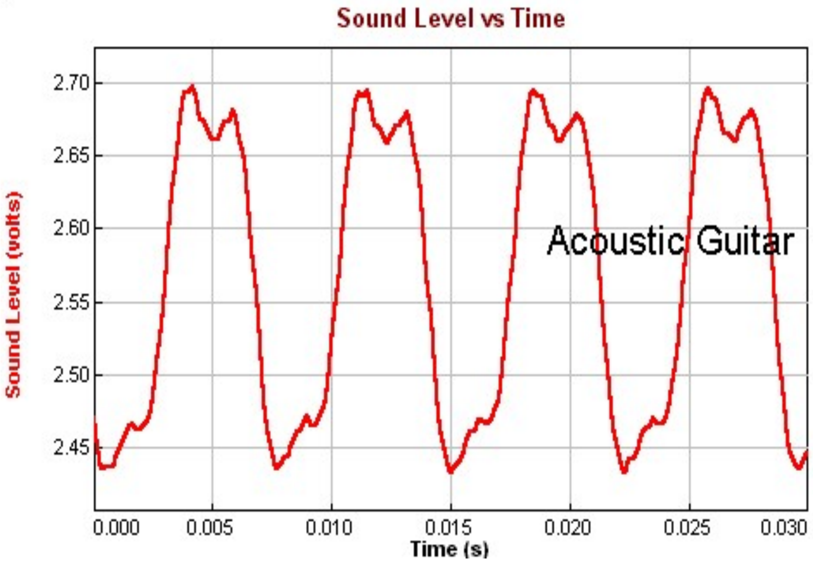
Characteristics of sound in audio signals:

- High pitch - rapidly varying signal
- Low pitch - slowly varying signal









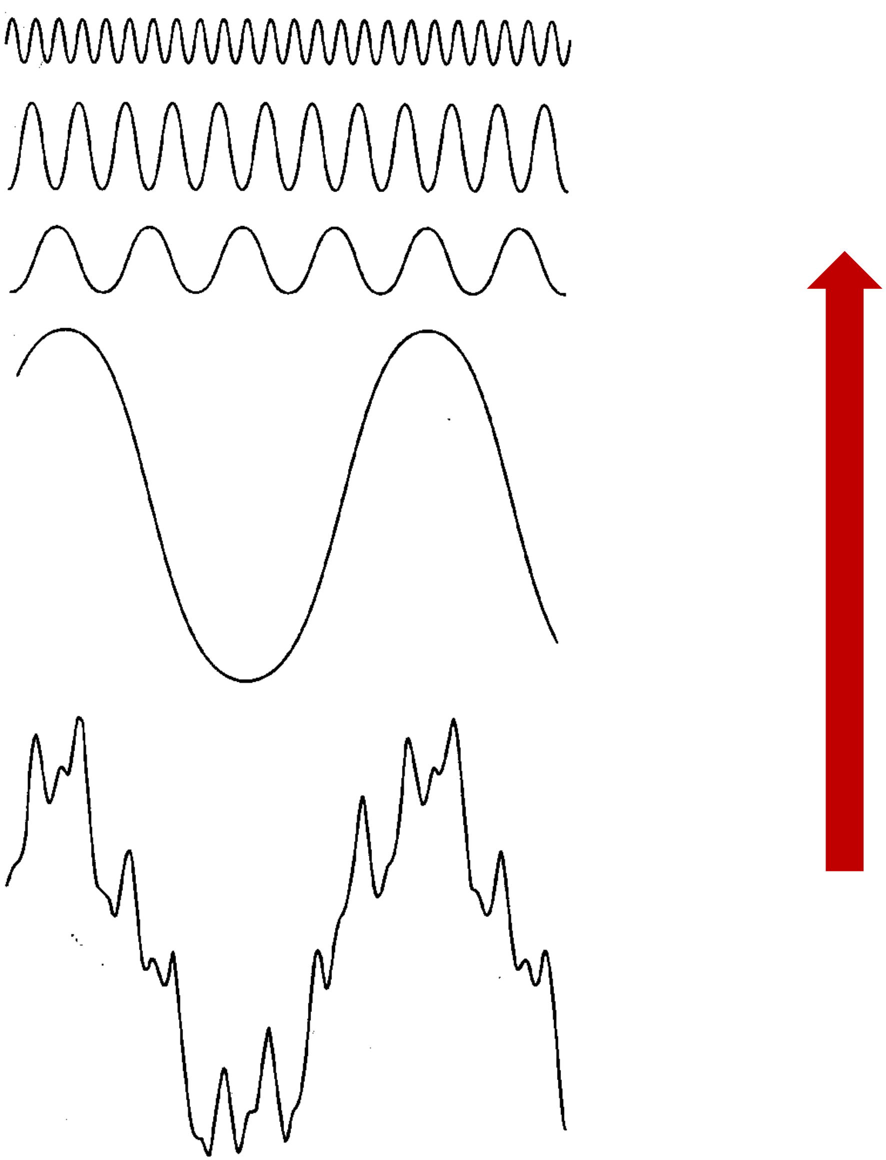
Frequency Analysis

Trigonometric Fourier Series: Any periodic function can be expressed as the sum of sines and/or cosines of different frequencies, each multiplied by a different coefficient. → *Jean Baptiste Joseph Fourier* (1822).

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) + b_n \sin\left(\frac{2\pi n x}{T}\right)$$

- A function with period T is represented by two infinite sequences of coefficients. n is the no. of cycles/period.
- The sines and cosines are the Basis Functions of this representation. a_n and b_n are the Fourier Coefficients.
- The sinusoids are harmonically related: each one's frequency is an integer multiple of the fundamental frequency of the input signal.

Expressing a periodic function as a sum of sinusoids



The Trigonometric Fourier Series

once more…

A trigonometric *Fourier series* is an expansion of a periodic function $f(x)$. This expansion is in terms of an infinite sum of sines and cosines.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{T}\right) \;+\; b_n \sin\left(\frac{2\pi nx}{T}\right)$$

cf. with slide 12 – when $n=0$ the sin term disappears and the cos term is 1, so we can rewrite the equation as above.

The Fourier series allows any arbitrary periodic function to be broken into a set of simple terms that can be solved individually, and then combined to obtain the solution to the original problem or an approximation to it.

a_0 is often referred to as the DC term or the average of the signal.

Fourier Series solution

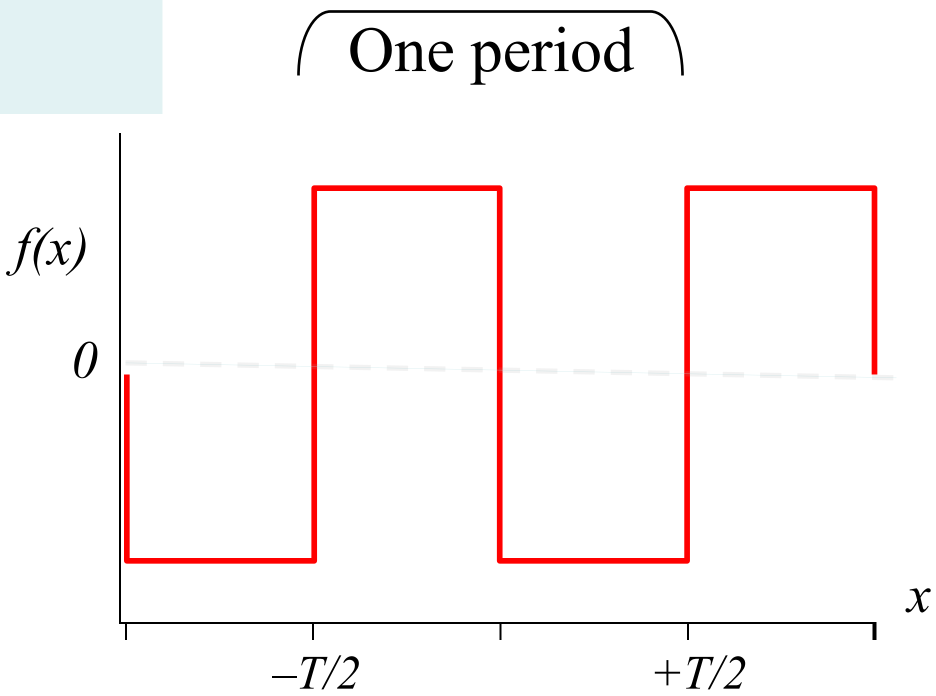
A *Fourier series* provides an equivalent representation of the function:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{T}\right) \; + \; b_n \sin\left(\frac{2\pi n x}{T}\right)$$

The coefficients are:

$$a_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \cos(\frac{2\pi n x}{T}) dx$$

$$b_n = \frac{2}{T} \int_{-T/2}^{+T/2} f(x) \sin(\frac{2\pi n x}{T}) dx$$



Example periodic function on *-T/2, +T/2*

Fourier Series Example: Square Wave

- $f(x)$ is a square wave

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\cos(2\pi nx/T)dx$$

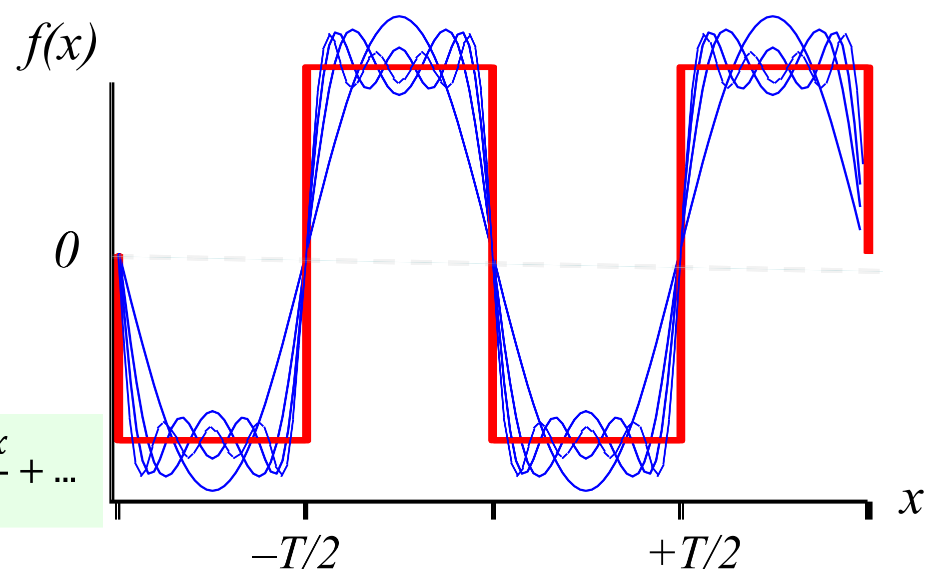
$$= \frac{2}{T}\int_{-T/2}^{0} \cos(2\pi nx/T)dx - \frac{1}{T}\int_{0}^{+T/2} \cos(2\pi nx/T)dx \quad = 0$$

$$f(x) = \begin{cases} +1 & \frac{-T}{2} \le x < 0 \\ -1 & 0 \le x < \frac{T}{2} \end{cases}$$

$$n = 1,3,5,7,\ldots$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} f(x)\sin(2\pi nx/T)dx$$

$$= \begin{cases} \dfrac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

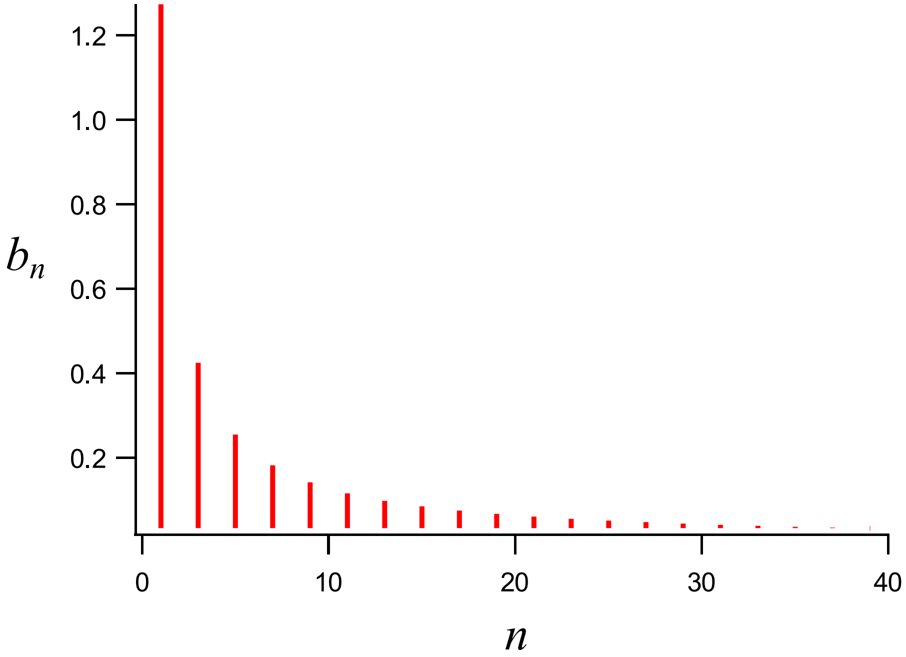


$$f(x) = \frac{4}{\pi}\cdot\sin\frac{2\pi x}{T} + \frac{4}{3\pi}\cdot\sin 3\cdot\frac{2\pi x}{T} + \frac{4}{5\pi}\cdot\sin 5\cdot\frac{2\pi x}{T} + \ldots$$

Fourier Series Example: Square Wave

- The set of *Fourier-space* coefficients b_n contain complete information about the function
- Although *f(x)* is periodic to infinity, b_n is negligible beyond a finite range

- Sometimes the Fourier representation is more convenient to use, or just view



Frequency Analysis

- The aim of processing a signal using Fourier analysis is to *manipulate the spectrum of a signal* rather than manipulating the signal itself.
- Example: simple compression

- Functions that are ***not periodic*** can also be expressed as the integral of sines and/or cosines weighted by a coefficient. In this case we have the Fourier transform.
- The Fourier transform provides a way of representing a signal in a different space - i.e., in the frequency domain.

Fourier Transform Applications

- Applications wide ranging and ever present in modern life:
  - *Telecomms/Electronics/IT* - cellular phones, digital cameras, satellites, etc.
  - *Entertainment* - music, audio, multimedia devices
  - *Industry* - X-ray spectrometry, Car ABS, chemical analysis, radar design
  - *Medical* - PET, CAT, & MRI machines
  - *Image and Speech analysis* (voice activated “devices”, biometry, …)
  - and many other fields...

1D Fourier Transform

- The Fourier Transform of a single variable continuous function $f(x)$ is:

$$F(u) = \int_{-\infty}^{\infty} f(x) \quad e^{-j2\pi ux} dx$$

- Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x) = \int_{-\infty}^{\infty} F(u) \quad e^{j2\pi ux} du$$

These two equations are also known as the Fourier Transform Pair.
Note, they constitute a lossless representation of data.

1D Fourier Transform: Discrete Form

- The Fourier Transform of a discrete function of one variable, $f(x),\ x=0,1,2\ldots,N\text{-}1$ is:

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)\,e^{\frac{-j2\pi ux}{N}}\qquad \text{for } u=0,1,2,\ldots,N-1.$$

- Conversely, given $F(u)$, we can obtain $f(x)$ by means of the *inverse* Fourier Transform:

$$f(x)=\sum_{u=0}^{N-1}F(u)\,e^{\frac{j2\pi ux}{N}}\qquad \text{for } x=0,1,2,\ldots,N-1.$$

These two equations are also known as the Fourier Transform Pair.
Note, they constitute a lossless representation of data.

1D Fourier Domain

- The concept of the frequency domain follows from Euler's Formula:

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

- Thus each term of the Fourier Transform is composed of the sum of *all* values of the function $f(x)$ multiplied by sines and cosines of various frequencies:

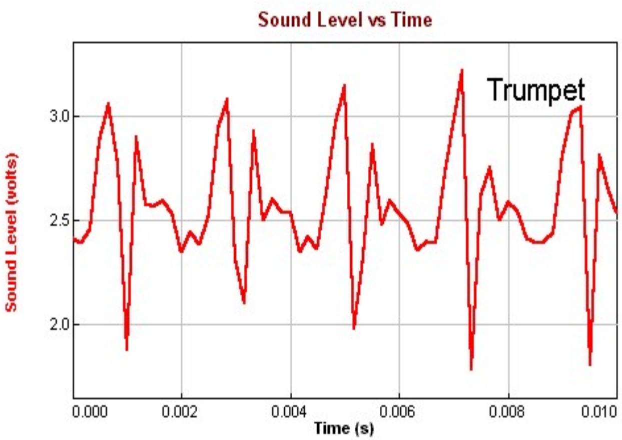
$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\left[\cos\left(\frac{2\pi ux}{N}\right) - j\sin\left(\frac{2\pi ux}{N}\right)\right]$$

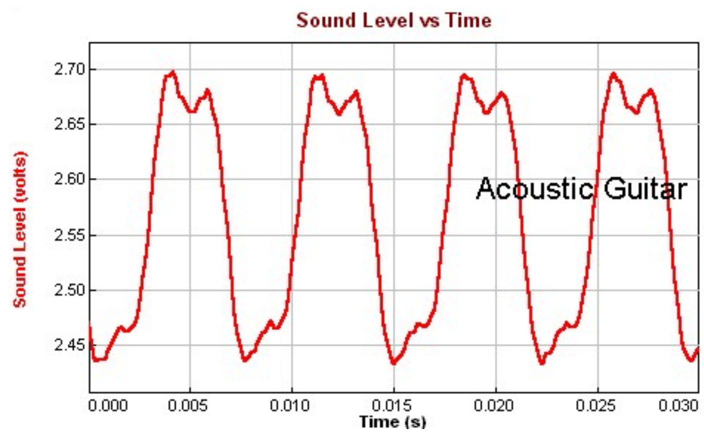
for $u = 0,1,2,\ldots,N-1.$

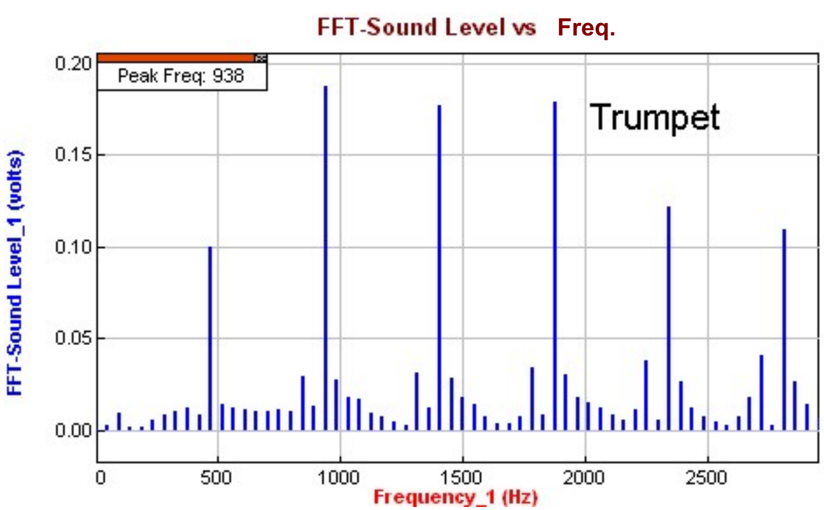
We have transformed from a time domain to a frequency domain representation.

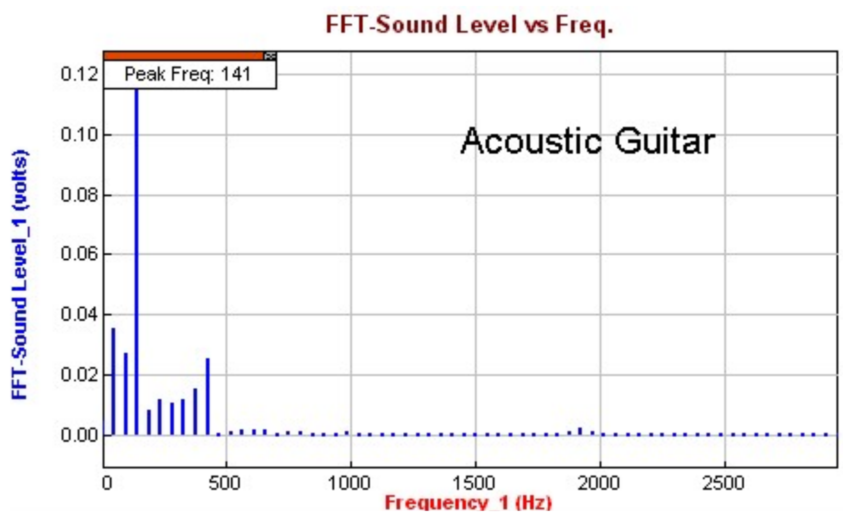
Example: Low and High Frequency

Characteristics of sound in audio signals.



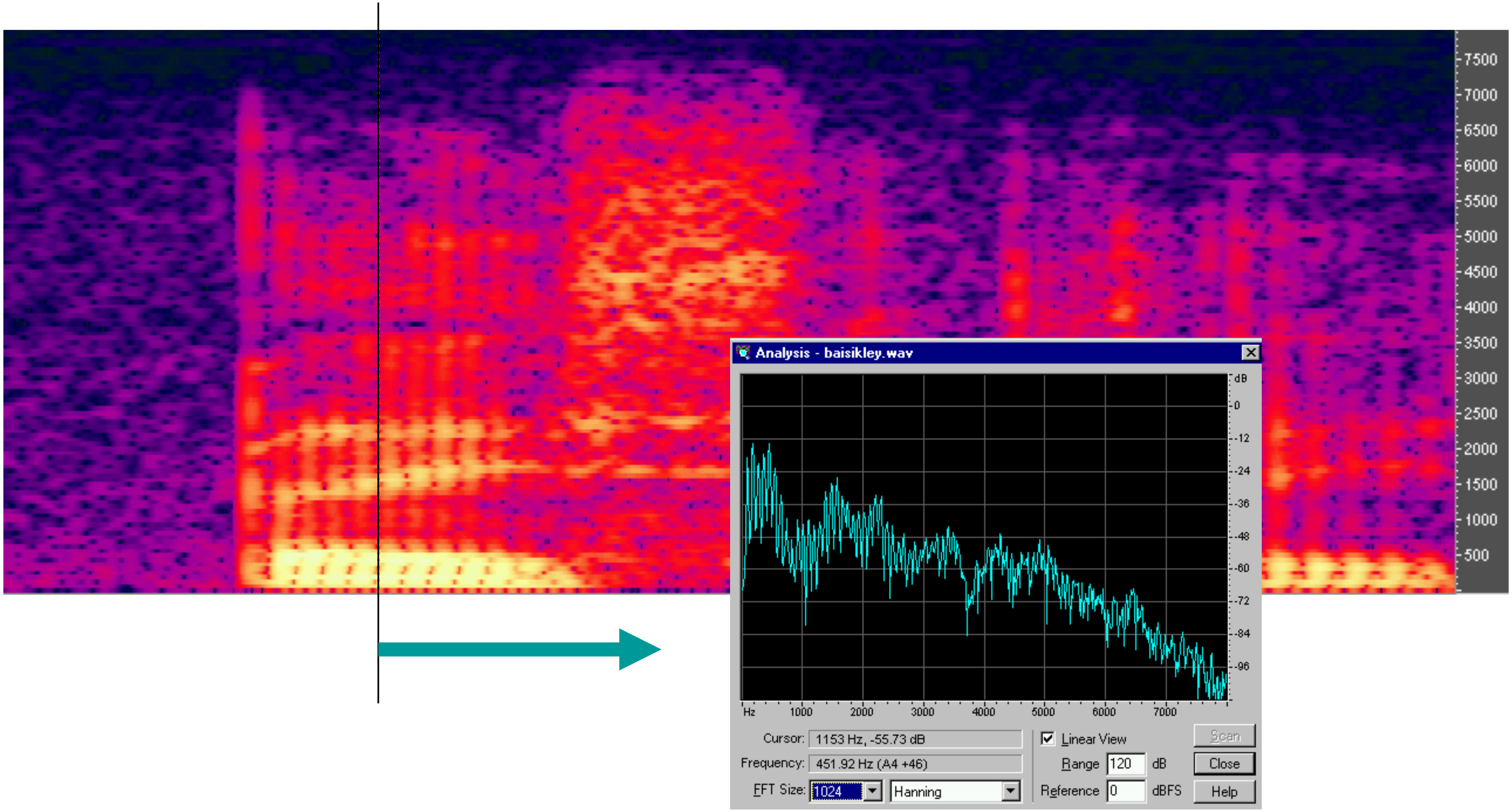






Example: Acoustic Data Analysis

Spectrogram



1D Fourier Transform

- $F(u)$ is a complex number & has real and imaginary parts:

$$F(u) = R(u) + jI(u)$$

- *Magnitude* or *spectrum* of the FT:

$$|F(u)| = \sqrt{R^2(u) + I^2(u)}$$

- Phase angle or phase spectrum:

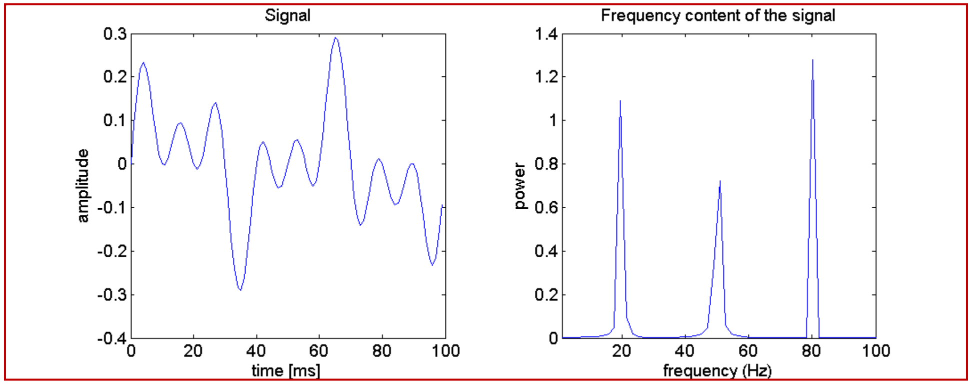
$$\varphi(u) = \tan^{-1}\frac{I(u)}{R(u)}$$

- Expressing $F(u)$ in polar coordinates:

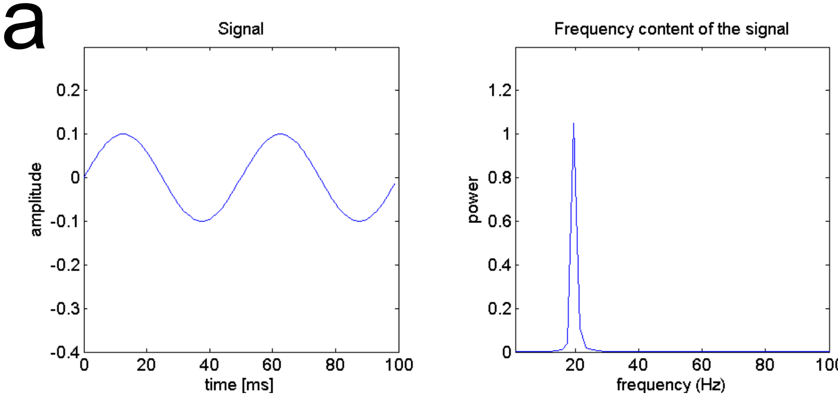
$$F(u) = |F(u)|e^{j\varphi(u)}$$

Simple 1D example

d = a + b + c



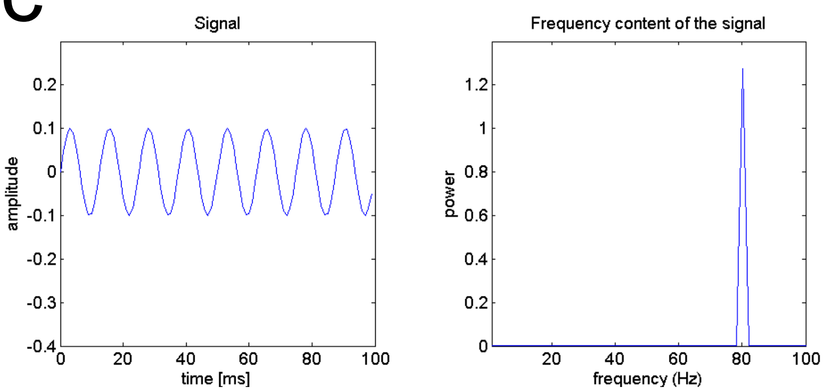
a



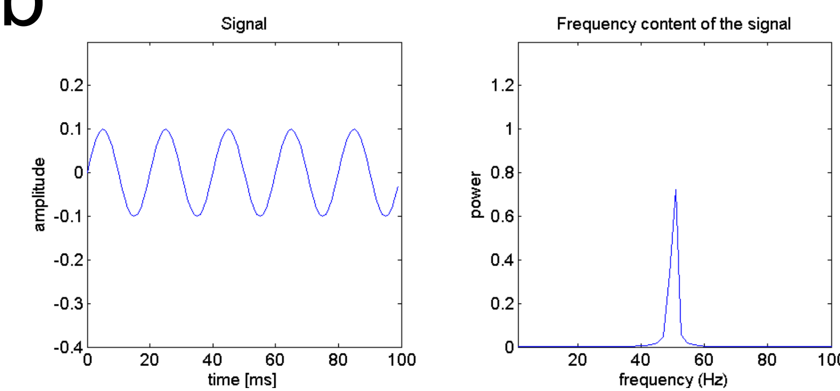
time domain frequency domain

time domain frequency domain

c



b



time domain frequency domain

time domain frequency domain

Frequency Spectrum

- Distribution of $|F(u)|$ → frequency spectrum of signal.
- Slowly changing signals → spectrum concentrated around low frequencies.
- Rapidly changing signals → spectrum concentrated around high frequencies.
- Hence low and high frequency signals.
- Also bandlimited signals → frequency content confined within some frequency band.

Very Simple Application example

- Automatic speech recognition between two speech utterances *x(n)* and *y(n)*.
- Naïve approach:

$$E = \sum_{\forall n} (x(n) - y(n))^2$$

Problems with this approach?

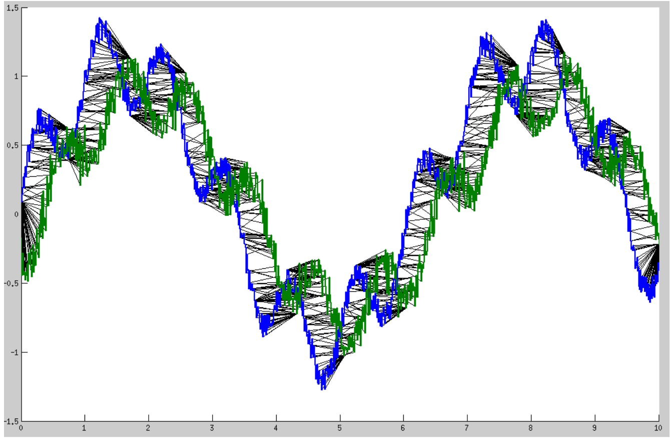
$x(n) = K\,y(n)$, yet $E \neq 0$

(K being a scaling parameter)

$x(n) = y(n-m)$, yet $E \neq 0$

(m causing a delay shift)

One solution could be Dynamic Time Warping (recall from earlier lecture)



Frequency domain features

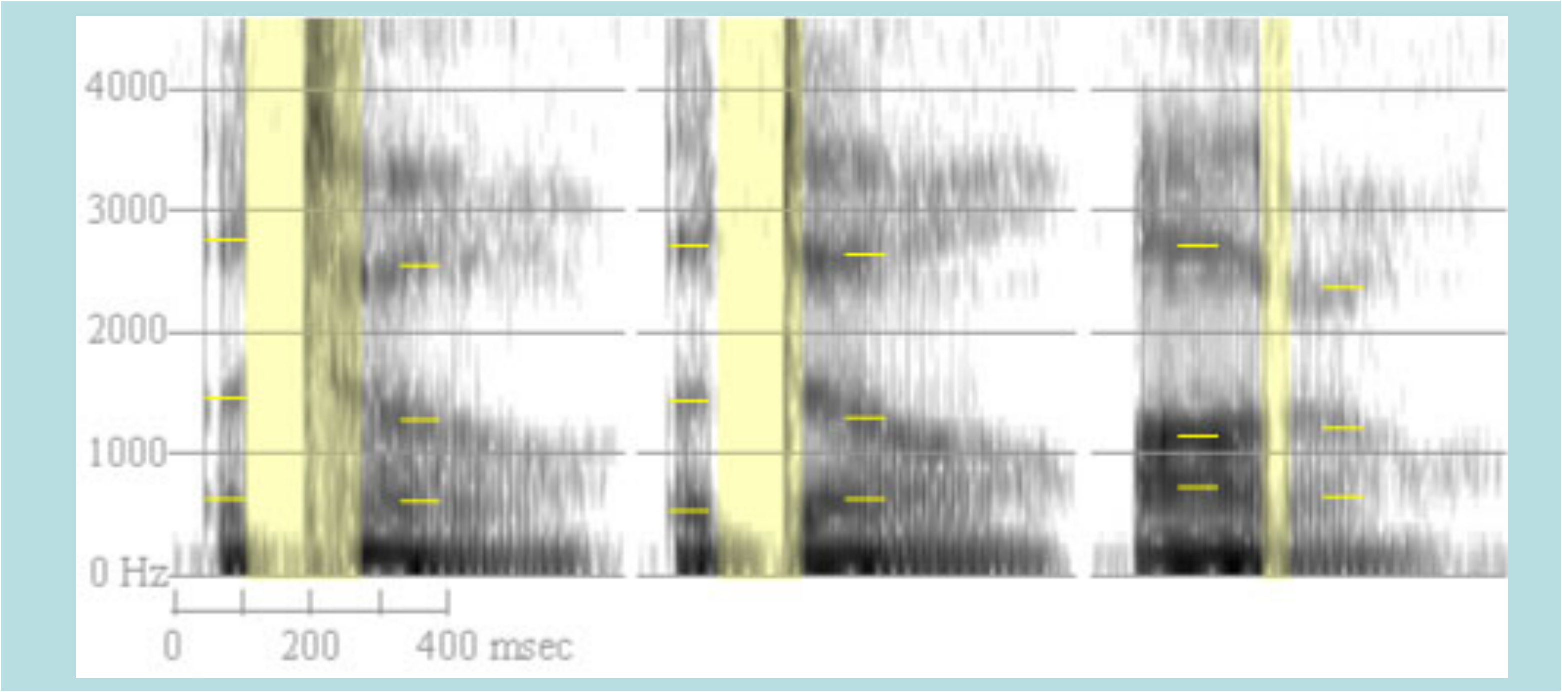
- Take the Fourier transform of both utterances to get $X(u)$ and $Y(u)$.

- Then consider the Euclidean distance between their magnitude spectrums: $|X(u)|$ and $|Y(u)|$:

$$d_E = \sum_{\forall u} (|X(u)| - |Y(u)|)^2$$

Frequency domain analysis

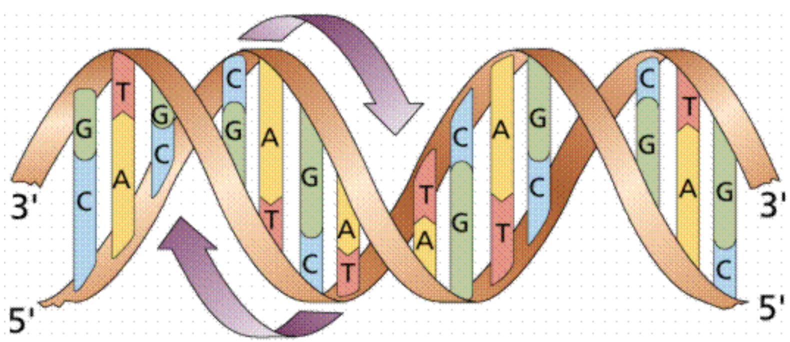
- Still a difficult task even in the frequency domain.



a tot *a dot* *otto*

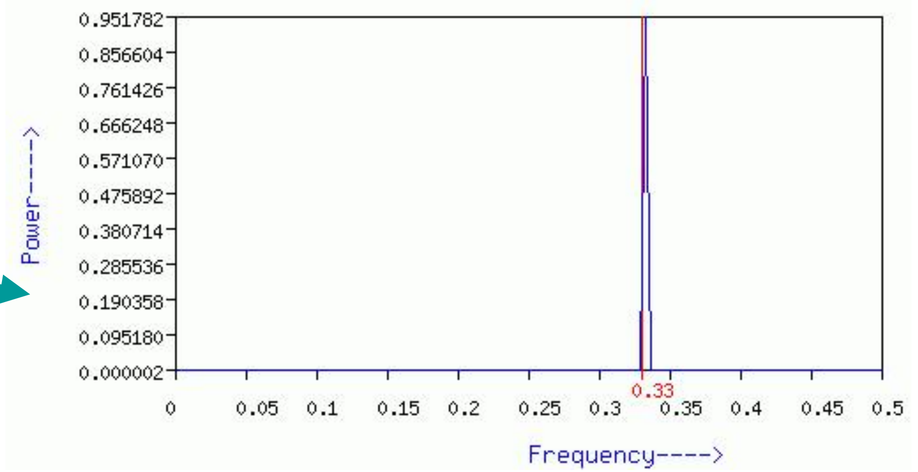
DNA sequence FT example

- The analysis of correlations in DNA sequences is used to identify protein coding genes in genomic DNA.
- Locating and characterizing repeats and periodic clusters provides certain information about the structural and functional characteristics of the molecule.
- DNA sequences are represented by letters, A, C, G or T, and - .
- e.g. ACAATG-GCCATAAT-ATGTGAAC--GCTCA…

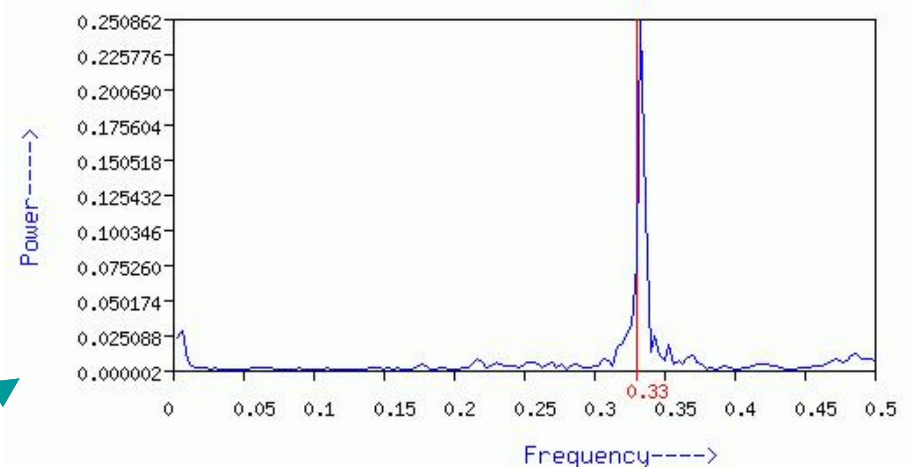


DNA sequence FT example

- Consider the periodic sequence A--A--A--A--..... where blanks can be filled randomly by A, C, G or T. This shows a periodicity of 3.



- The spectral density of such a sequence is significantly non-zero only at one frequency (0.33) which corresponds to the perfect periodicity of base A (1/0.333=3.0).



- Destroy the perfect repetition by randomly replacing the A's with all letters…

Let's practice: DNA sequence analysis

- The computation of Fourier and other linear transforms of *symbolic data* is a big problem.
- The simplest solution is to map each symbol to a number. The difficulty with this approach is the dependence on the particular labeling adopted.

Consider, for example, the following symbolic periodic sequence:
s = (ATAGACATAGAC . . .).

The mapping		
A → 1, T → 0, G → 0, C → 0,	→	Period Two

The mapping		
A → 1, T → 2, G → 3, C → 4,	→	Period Six

- This clearly shows that some of the relevant harmonic structure can be exposed by the symbolic-to-numeric labelling.